SHGH Conjecture and unexpected curves

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Joint work with

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Outline

Part 1

Introduction to the SHGH Conjecture and the unexpected curves

Notations

- $\mathbb{K}$ will denote an infinite field of arbitrary characteristic.
- Our polynomial ring will be $R = \mathbb{K}[x, y, z]$.
- $\mathscr{L}_{2,d}$ will denote the linear system of plane curves of degree d in $\mathbb{P}^2 = \mathbb{P}^2_{\mathbb{K}}$.
- Given a fat point subscheme $X = m_1 P_1 + \cdots + m_r P_r$ of $\mathbb{P}^2$ with distinct points P_i,

  $$\mathscr{L}_{2,d,X}$$

  will denote the linear system of curves of degree d in $\mathbb{P}^2$ containing X, that is, having multiplicity at least m_i at P_i for $i = 1, \ldots, r$. Instead, when the P_i's are supposed **general**, we will use the notation

  $$\mathscr{L}_{2,d}(m_1^{l_1}, \ldots, m_t^{l_t})$$

  where $l_1 + \cdots + l_t = r$ and m_i is repeated l_i times.

Special and non-special linear systems

Definition 1.1

Given a fat point subscheme $X = m_1P_1 + \ldots + m_rP_r$ with distinct points P_i, the **expected dimension** of $\mathscr{L}_{2,d,X}$ is

$$\operatorname{expdim}(\mathscr{L}_{2,d,X}) \doteq \max\left\{\dim(\mathscr{L}_{2,d}) - \sum_{i=1}^{r}\binom{m_i+1}{2}, -1\right\}.$$

In general

$$\dim(\mathscr{L}_{2,d,X}) \geq \operatorname{expdim}(\mathscr{L}_{2,d,X}). \tag{1}$$

Definition 1.2

The system $\mathscr{L}_{2,d,X}$ is said to be **non-special** if the equality holds in (1). Otherwise it is said to be **special**.

The same definitions hold for $\mathscr{L}_{2,d}(m_1^{l_1}, \ldots, m_t^{l_t})$ too.

The SHGH Conjecture

Conjecture 1.3 (B. Segre, 1961, [Seg61])

If the linear system $\mathscr{L}_{2,d}(m_1^{l_1}, \ldots, m_t^{l_t})$ is special, then its general member is non reduced, namely the linear system has, according to Bertini's Theorem, some multiple fixed component.

There are other versions for the above conjecture:

- Harbourne (1986) [Har86]
- Gimigliano (1987) [Gim87]
- Hirschowitz (1989) [Hir89]

Ciliberto and Miranda in 2001 [CM01] proved that the last three versions are all equivalent to the first one.

A list of known cases

The SHGH Conjecture is true for the following linear systems:

- All linear systems $\mathscr{L}_{2,d}(m_1^{l_1}, \ldots, m_t^{l_t})$ with $l_1 + \ldots + l_t \leq 9$ (Castelnuovo, 1891; Nagata, 1960; Gimigliano, Harbourne, 1986).
- $\mathscr{L}_{2,d}(m^l)$ (homogeneous case), $m \leq 3$ (Hirschowitz, 1985).
- $\mathscr{L}_{2,d}(m_1, m_2^{l_2})$ (quasi-homogeneous case), $m_2 \leq 3$ (Ciliberto-Miranda, 1998).
- $\mathscr{L}_{2,d}(m^l)$, $m \leq 12$ (Ciliberto-Miranda, 1998).
- $\mathscr{L}_{2,d}(m, 4^l)$ (Laface, 1998).
- and other more special cases...

Beyond the SHGH Conjecture

Further research in two different directions:

- **Consider in general $\mathbb{P}^n$ but assign low multiplicities m_i's.**
  Alexander and Hirschowitz in 1995 solved the problem in the case $m_i = 2\ \forall i$ and confirmed the SHGH Conjecture!

- **Replace $\mathscr{L}_{2,d}$ with a suitable subsystem.**

Cook II, Harbourne, Migliore and Nagel in [Coo+16] took the second direction.

In particular they considered an additional fat point subscheme $Z = c_1 Q_1 + \cdots + c_s Q_s$ with distinct points Q_i, and the linear system $\mathscr{L}_{2,d,Z}$ of plane curves of degree d containing Z.
Then they considered the linear system $\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})$ and looked for speciality in this context.

Beyond the SHGH Conjecture

In this generalized context, we have this definition.

Definition 1.4

Given fat point subschemes $Z = c_1 Q_1 + \cdots + c_s Q_s$ with distinct points Q_i and $X = m_1 P_1 + \cdots + m_r P_r$ with general distinct points P_i of $\mathbb{P}^2$, the **expected dimension** of $\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})$ is

$$\operatorname{expdim}\left(\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})\right) \doteq \max\left\{\dim(\mathscr{L}_{2,d,Z}) - \sum_{i=1}^{t} l_i \binom{m_i+1}{2}, -1\right\}.$$

Again we have that

$$\dim\left(\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})\right) \geq \operatorname{expdim}\left(\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})\right). \tag{2}$$

Note that there is a difference between $\operatorname{expdim}\mathscr{L}_{2,d,Z}(X)$ and $\operatorname{expdim}\mathscr{L}_{2,d}(Y)$ where $Y = Z + X$! We are interested in the first case, that is, we are not interested in the number of conditions that Z imposes on the curves of degree d.

Speciality and Unexpected Curves

Definition 1.5

The system $\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})$ is said to be **non-special** if the equality holds in (2). Otherwise it is said to be **special**. In particular, if $\operatorname{expdim}\left(\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})\right) = -1$ but in fact $\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t}) \neq \emptyset$, we say that Z admits an **unexpected curve of degree d with respect to X**.

Therefore the new problem posed in [Coo+16] is the following:

Open Problem

Given fat point subschemes $Z = c_1 Q_1 + \cdots + c_s Q_s$ with distinct points Q_i and $X = m_1 P_1 + \cdots + m_r P_r$ with general distinct points P_i of $\mathbb{P}^2$, characterize and then classify all special linear systems $\mathscr{L}_{2,d,Z}(m_1^{l_1}, \ldots, m_t^{l_t})$. In particular, classify all Z's which admit an unexpected curve of degree d with respect to X.

A very special context

Previous open problem in this generality is currently inaccessible.
Therefore the authors of [Coo+16] considered a very special setting, the one in which:

- Z is reduced (that is, $Z = Q_1 + \ldots + Q_s$ for distinct points Q_i), of degree at least 2 ($|Z| \geq 2$).
- $t = 1$ (homogeneous case).
- $m_1 = d - 1$.

Summing up, the problem we're focusing now is the following:

Problem

Characterize and classify all the fat point subschemes $Z = c_1 Q_1 + \cdots + c_s Q_s$ with distinct points Q_i which admit an unexpected curve of degree d with respect to $X = (d-1)P$.

The invariants m_Z, t_Z and u_Z

Definition 1.6 (Definition 4.1 of [FV14])

Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$. We define the **multiplicity index** m_Z to be

$$m_Z = \min\{d \mid \mathscr{L}_{2,d,Z}(d-1)) \neq \emptyset\}$$

Definition 1.7 (Definition 2.5 of [Coo+16])

Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$. We define

$$t_Z = \min\left\{d \mid \dim(\mathscr{L}_{2,d,Z}) \geq \binom{d}{2}\right\}.$$

Definition 1.8 (Definition 3.1 of [Coo+16])

Let Z as before and $P \in \mathbb{P}^2$ be a general point. We define the **speciality index** u_Z to be

$$u_Z = \min\{d \mid Z + (d-1)P \text{ imposes independent conditions on plane curves of degree } d\}.$$

Main result of [Coo+16]

Theorem 1.9 (Theorem 3.7 of [Coo+16])

Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$. Denote with $h_Z(\cdot)$ its Hilbert function. If $h_Z(t_Z) < |Z|$, then Z admits no unexpected curves.

Theorem 1.10 (Theorem 3.9 of [Coo+16])

Let Z be a reduced 0-dimensional subscheme of $\mathbb{P}^2$. Then Z admits an unexpected curve if and only if $m_Z < t_Z$. Furthermore, in this case Z has an unexpected curve of degree d if and only if $m_Z \leq d - 1 < u_Z$.

The following is a more geometric version of Theorem 1.10.

Corollary 1.11 (Corollary 5.7 of [Coo+16])

Let $Z \subseteq \mathbb{P}^2$ be a finite set of points. Then Z admits an unexpected curve if and only if $2m_Z + 2 < |Z|$ but no subset of $m_Z + 2$ (or more) of the points is collinear. In this case, Z has an unexpected curve of degree d if and only if $m_Z < d \leq |Z| - m_Z - 2$.

Further results of [Coo+16]

On one hand, the presence of unexpected curves forces Z not to have too much collinear points.
On the other hand, as we see from next result, the points of Z cannot be linearly general.

Corollary 1.12 (Corollary 6.8 of [Coo+16])

Let Z be a set of points of $\mathbb{P}^2$ in linear general position. Then $m_Z = \lfloor \frac{|Z|-1}{2} \rfloor$, $u_Z = \lceil \frac{|Z|-1}{2} \rceil - 1$, and Z does not admit an unexpected curve. Furthermore, for a general point P,

$$\dim(\mathscr{L}_{2,m_Z+1,Z}(m_Z)) = \begin{cases} 1 & \text{if } |Z| \text{ is odd;} \\ 0 & \text{if } |Z| \text{ is even;} \end{cases}$$

and $\mathscr{L}_{2,m_Z+1,Z}(m_Z)$ contains an irreducible form.

Further results of [Coo+16]

Another question answered in [Coo+16] is related with the irreducibility of unexpected curves and, if they are reducible, what do the irreducible components look like.

Lemma 1.13 (Lemma 5.1 of [Coo+16])

Let Z be a finite set of points of $\mathbb{P}^2$, and let $P \in \mathbb{P}^2$ be a general point. If C is a curve of degree $m_Z + 1$ containing Z, with multiplicity m_Z at a general point P (that is, an element of $\mathscr{L}_{2,m_Z+1,Z}(m_Z)$), then it is reduced and either it is irreducible, or it is a union of lines through P and an irreducible curve C' whose multiplicity at P is $\deg(C') - 1$. The curve C' is rational and smooth away from P.
Furthermore, the set $Z' = Z \cap C'$ has multiplicity index $m_{Z'} = m_Z - |Z''|$, where $Z'' = Z - Z'$, and each of the components of C other than C' passes through exactly one of the points of Z''. In particular,
$\deg(C') = \deg(C) - |Z''| = m_{Z'} + 1$.

Further results of [Coo+16]

Corollary 1.14 (Corollary 5.3 of [Coo+16])

Let Z be a finite set of points with an unexpected curve. Then Z has a unique unexpected curve C in degree $m_Z + 1$, and for each $m_Z + 1 < t \leq u_Z$, the unexpected curves of degree t are precisely the curves $C + L_1 + \cdots L_r$, where $r = t - m_Z - 1$ and each L_i is an arbitrary line through the point P (i.e., the general point at which C is singular).

What if we look for unexpected curves of degree d with respect to $X = (d-2)P$?

Previous result is not true any more...

The methods used

The main results of [Coo+16] apply different notions, like

- line arrangements dual to the configuration Z.
- the splitting type of a vector bundle, used already in [FV14].
- properties of Hilbert functions.

Moreover they use mathematical software like *Macaulay2* [GS02] and *CoCoA* [CoC08].

The dual line arrangement of Z

Let Z be a finite set of points of $\mathbb{P}^2$.
To each point $P = [p_1, p_2, p_3]$ of Z we associate a dual linear form $l_P = p_0x + p_1y + p_2z$ and a line L_P defined by the vanishing of l_P.
The union $\mathscr{A}_Z$ of these lines L_P for $P \in Z$ is called the **dual line arrangement** of Z.
We also associate to Z the homogeneous form $Q_Z = \prod_{P \in Z} l_P$. As a variety $\mathscr{A}_Z$ is defined by the vanishing of Q_Z.

The splitting type of $\mathcal{D}_Z$

Associated to any line arrangement $\mathscr{A}_Z$ is a locally free sheaf $\mathcal{D}_Z$ of rank 2, called the **derivation bundle** of Z.
More precisely, consider the submodule $D(Z) \subseteq R\frac{\partial}{\partial x} \oplus R\frac{\partial}{\partial y} \oplus R\frac{\partial}{\partial z} \cong R^3$ of $\mathbb{K}$-linear derivations δ such that $\delta(Q_Z) \in RQ_Z$. In particular, $D(Z)$ contains the Euler derivation $\delta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, and δ_E generates a submodule $R\delta_E \cong R(-1)$.
Then the derivation bundle $\mathcal{D}_Z$ of Z is the sheafification of the quotient $D(Z)/R\delta_E$.
When restricted to a general line L, $\mathcal{D}_Z$ splits as

$$\mathcal{O}_L(-a_Z) \oplus \mathcal{O}_L(-b_Z)$$

with $a_Z + b_Z = |Z| - 1$. The pair (a_Z, b_Z), where $a_Z \leq b_Z$, is called the **splitting type** of $\mathcal{D}_Z$ or $\mathscr{A}$.

- In [FV14] it is shown that $a_Z = m_Z$.
- Moreover in [Coo+16] it is proved that $b_Z = u_Z + 1$.

This allows to translate the results about finite sets of points into statements on line arrangements, and vice versa.

Part 2

Examples of unexpected curves in various degrees

An example of Z admitting an unexpected curve for $d = 3$

We reprise an example of Serre (see [Har77], Exercise III.10.7)

Example

Suppose that $char(\mathbb{K}) = 2$. Let Z be the seven points of $\mathbb{P}^2$ (called in this case *Fano plane*). Consider $d = 3$. Then $\dim(\mathscr{L}_{2,3,Z}) = 2$ and $\text{expdim}(\mathscr{L}_{2,3,Z}(2)) = -1$, so we expect there not to be an unexpected cubic. But

$$f = \alpha^2 yz(y+z) + \beta^2 xz(x+z) + \gamma^2 xy(x+y)$$

defines a (reduced) curve C which is singular at the general point $P = [\alpha, \beta, \gamma]$, therefore C is an unexpected cubic for Z.

Akesseh proved in 2017 in [Ake17] that this is the only one example of unexpected curve in degree 3.

It can be proved that if $char(\mathbb{K}) = 0$ there cannot exist an unexpected curve of degree 3 by using a non-defectivity argument.

An example of Z admitting an unexpected curve for $d = 4$

Proposition 2.1

The following configuration $Z = P_1 + \cdots + P_9$ of $\mathbb{P}^2$ admits an unexpected curve of degree 4:

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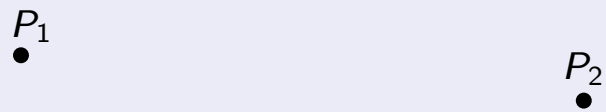

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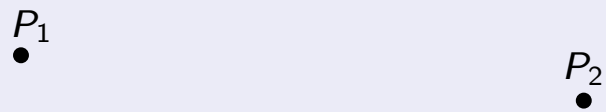

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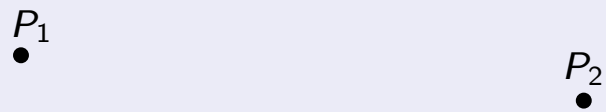

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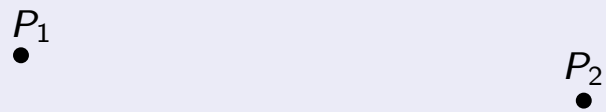

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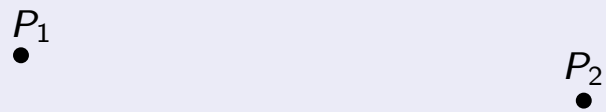

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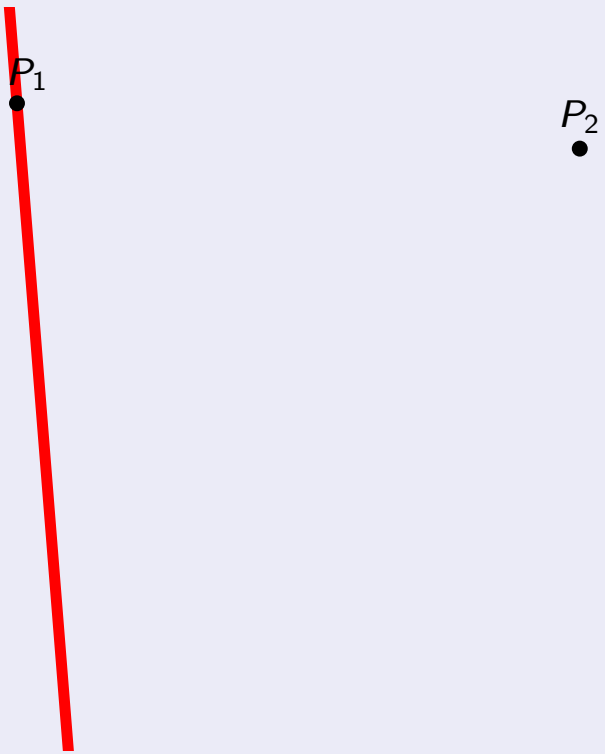

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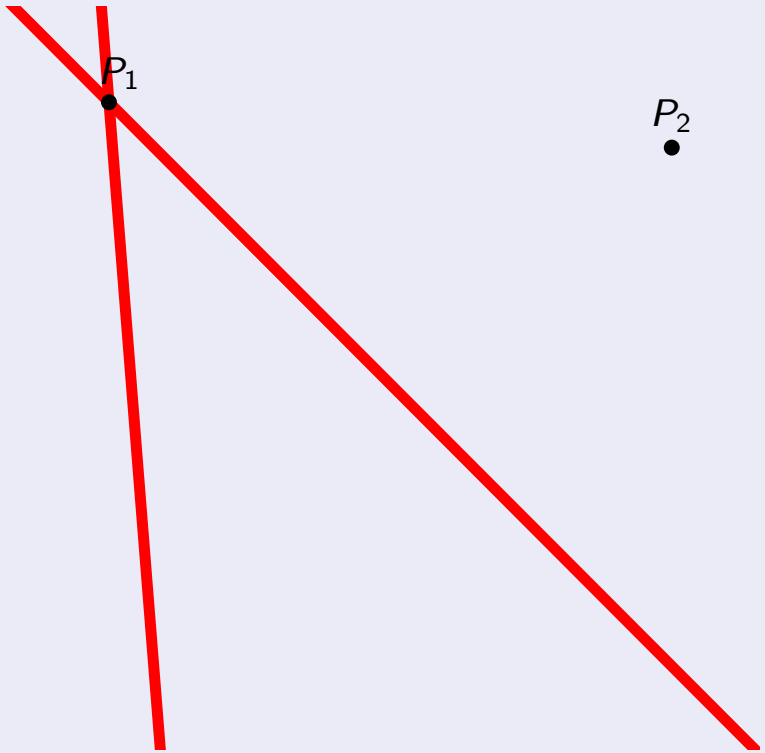

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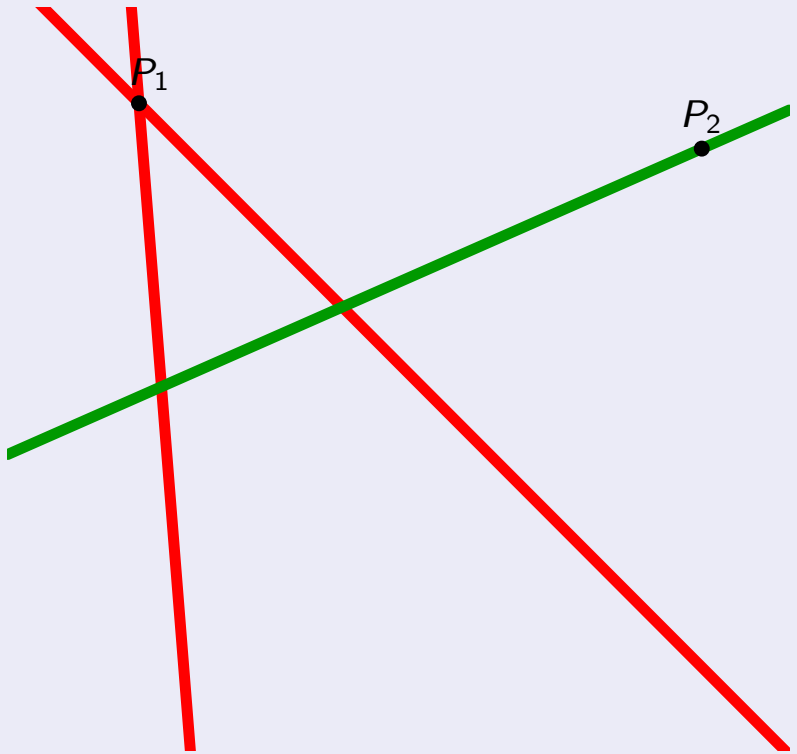

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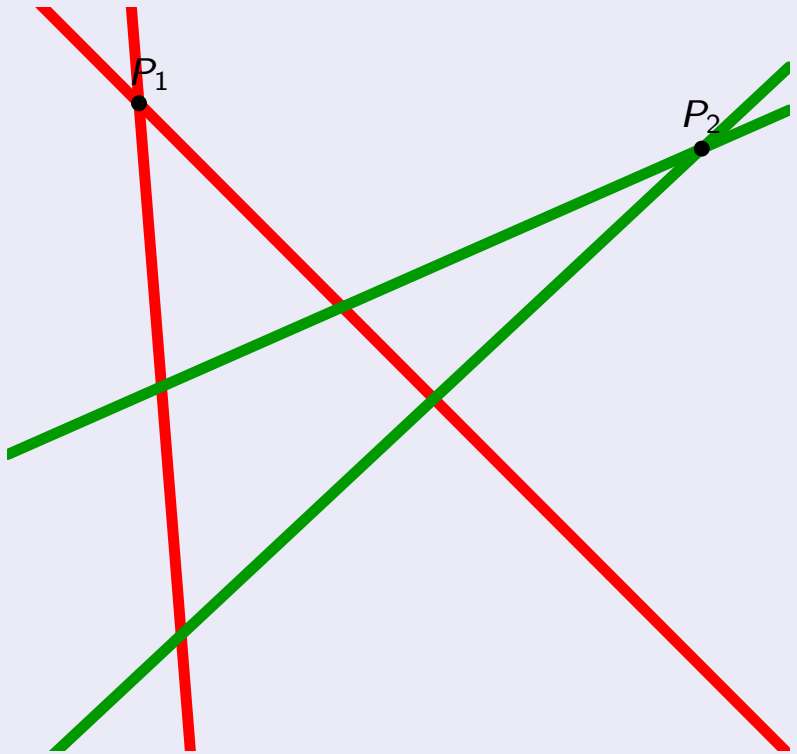

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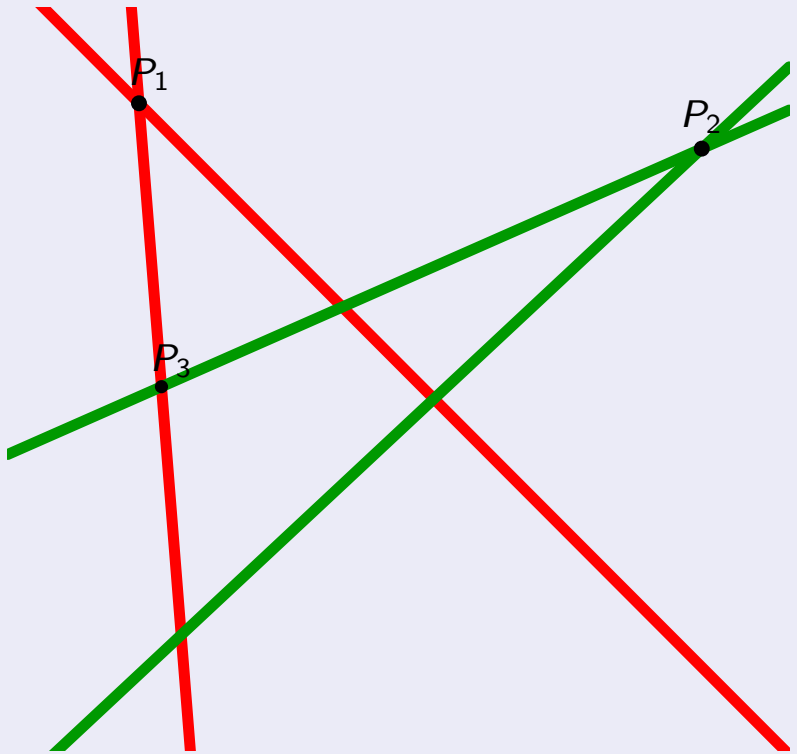

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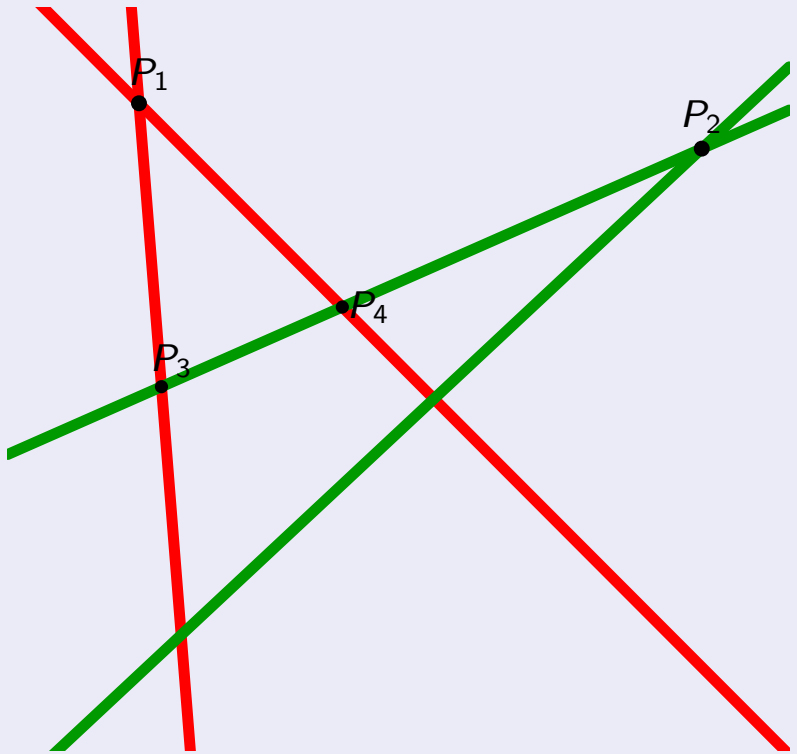

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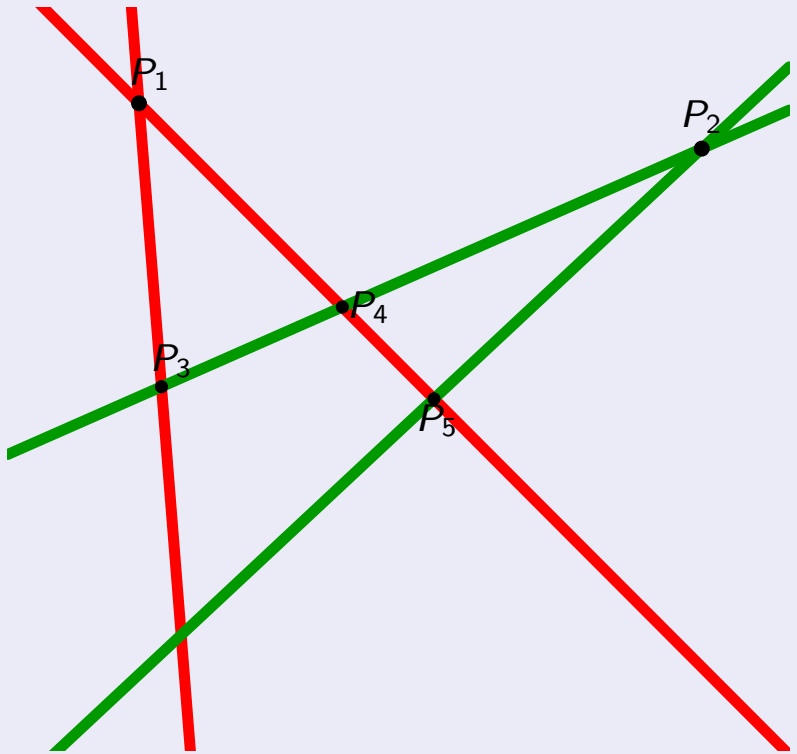

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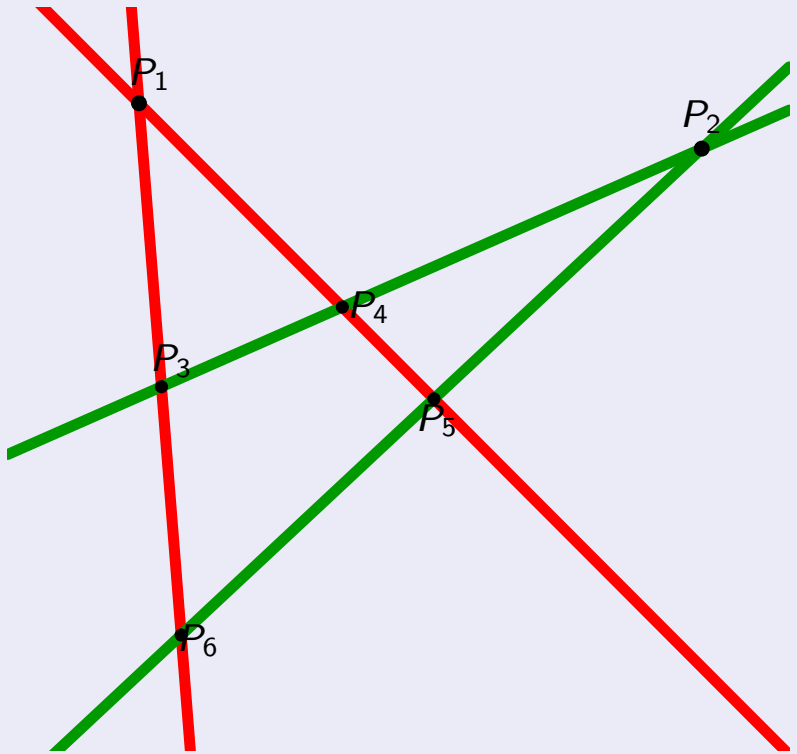

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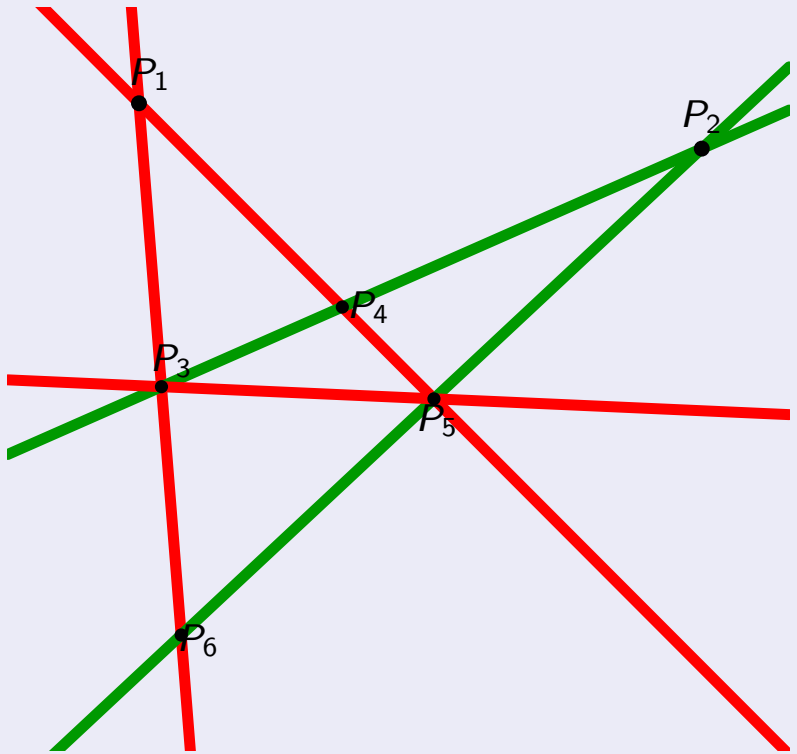

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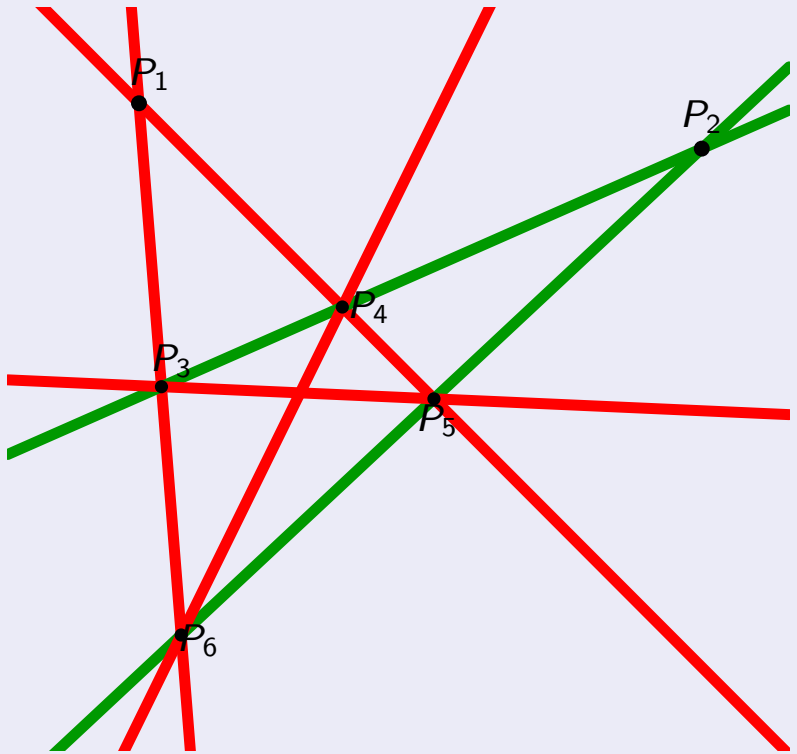

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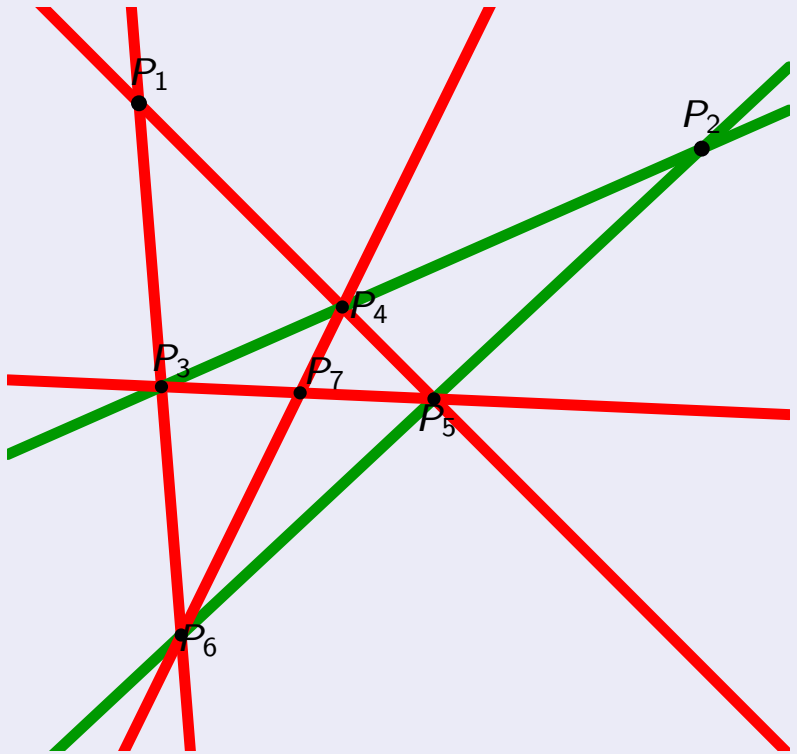

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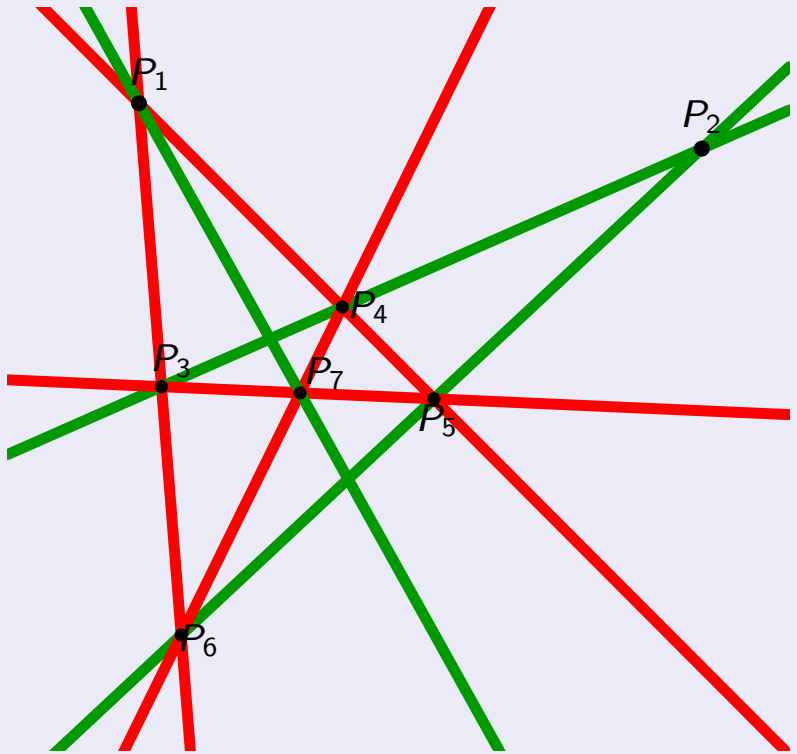

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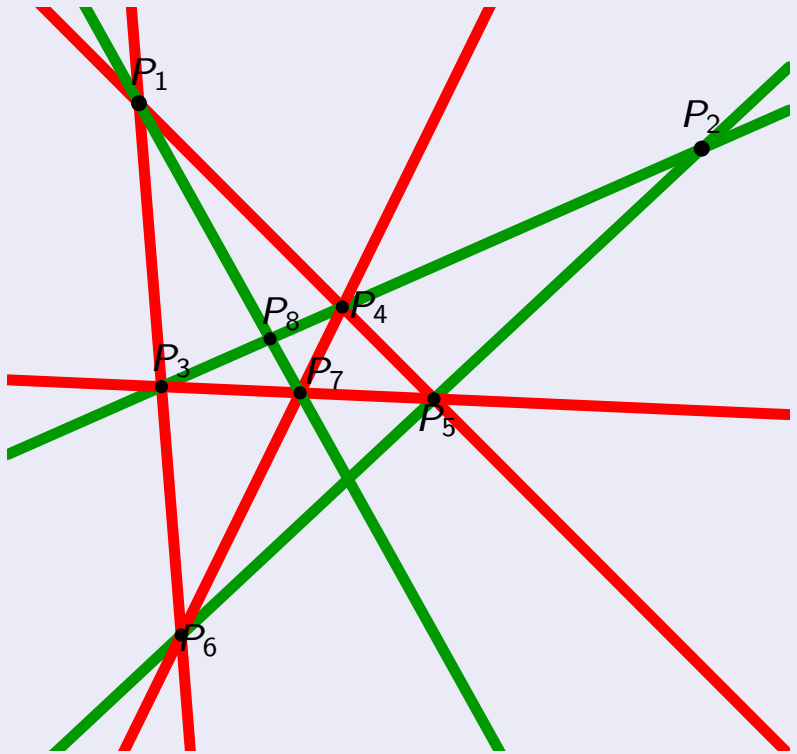
The following configuration $Z = P_1 + \cdots + P_9$ of $\mathbb{P}^2$ admits an unexpected curve of degree 4:



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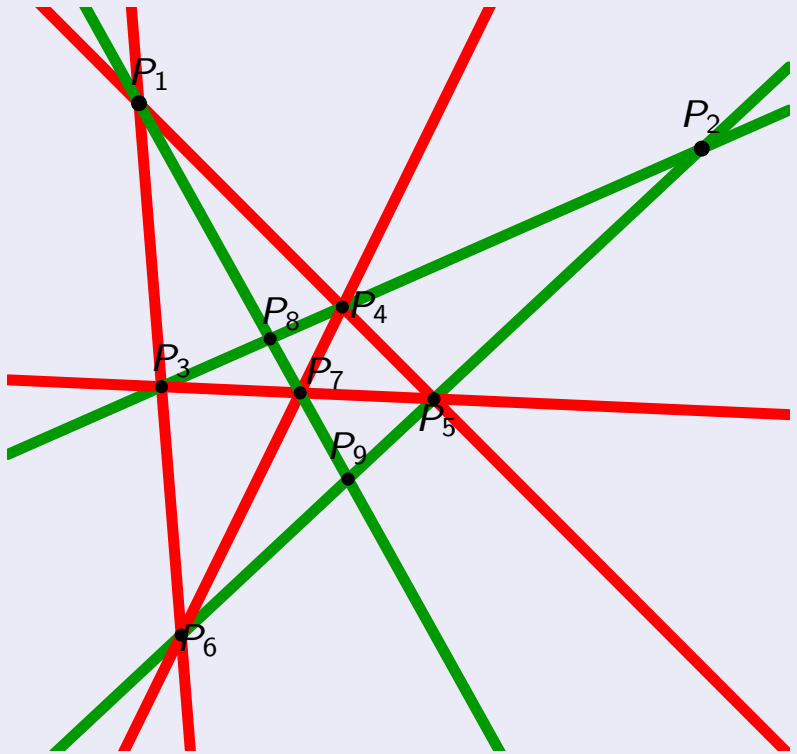


An example of Z admitting an unexpected curve for $d = 4$

Proposition 2.2

The reduced scheme Z of Proposition 2.1 is the only one (up to projectivity) which admits an unexpected curve of degree 4.

In the proof we used the following definition.

Definition 2.3

Consider a reduced 0-dimensional scheme Z and a line L of $\mathbb{P}^2$. We say that L is **rich** if it contains more than two points of Z. Moreover, given an integer $k \geq 0$, we say that L is k-**rich** if it contains exactly k points of Z.

Main steps of the proof:

0) If Z admits an unexpected quartic, then there cannot be a 5-rich line ($m_Z = 3$).
1) If Z admits an unexpected quartic and there exists a 4-rich line, then Z is the configuration of Proposition 2.1.
2) If Z admits an unexpected quartic, then there exists a 4-rich line.

About the proof of Proposition 2.2

Other remarks:

- In order to find a 4-rich line we focused on the number of 3-rich lines.
- We proved that the points of Z are not special with respect to conics (there are no 6 on a smooth conic).

Methods we used:

- a "specialization method", in special cases.
- application of the Bezout's Theorem.
- some of the proofs are Macaulay2-based.

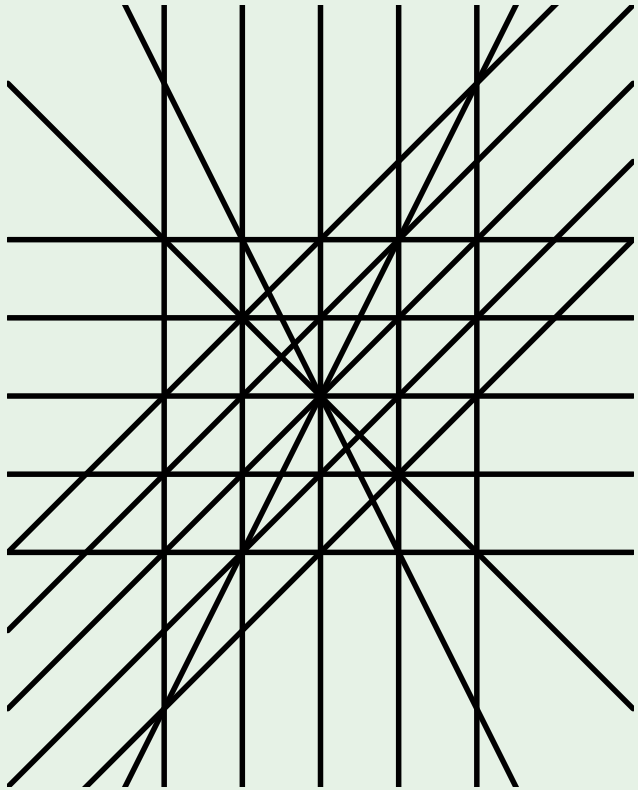
Pros and cons of our proof:

- Uses basic properties of plane projective geometry, it is a proof "by hand".
- This proof is too long and hard to generalize.

An example of not irreducible unexpected curve

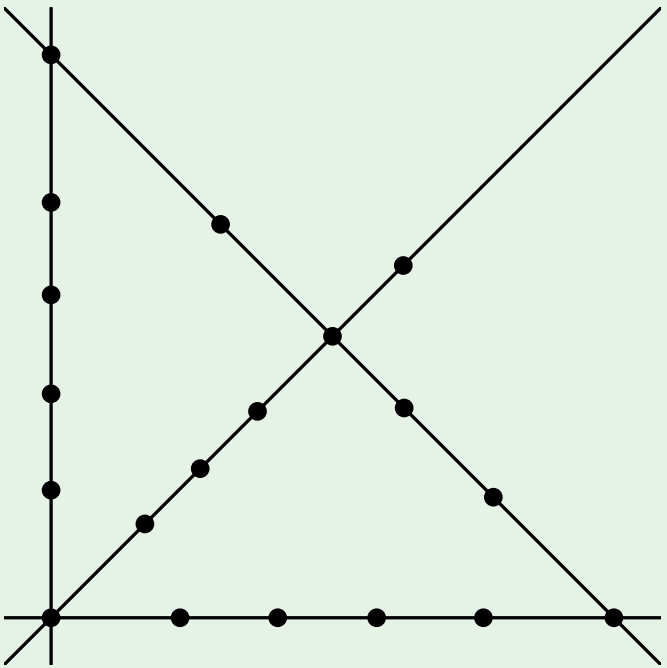
[Coo+16, Example 6.1]

For this example we assume that $\mathbb{K}$ has characteristic 0. Consider the line configuration given by the lines defined by the following 19 linear forms: x, y, z, $x+y$, $x-y$, $2x+y$, $2x-y$, $x+z$, $x-z$, $y+z$, $y-z$, $x+2z$, $x-2z$, $y+2z$, $y-2z$, $x-y+z$, $x-y-z$, $x-y+2z$, $x-y-2z$, shown in the following picture.



An example of not irreducible unexpected curve

Le Z be the corresponding reduced scheme consisting of the 19 points dual to the lines, sketched in the following picture. Using computer algebra systems, one can obtain the splitting type of Z and verify that Z admits an unexpected curve in degree 9. Moreover it can be shown that this unexpected curve is not irreducible, and indeed has two components, one of which is a line.



Examples of unexpected curves in higher degrees

A_Z	a_Z	b_Z
$A(13,2)$	5	7
$A(13,3)$	5	7
$A(17,2)$	7	9
$A(17,4)$	7	9
$A(17,3)$	7	9
$A(19,1)$	7	11
$A(19,3)$	7	11

A_Z	a_Z	b_Z
$A(20,5)$	8	11
$A(21,3)$	9	11
$A(21,4)$	9	11
$A(21,5)$	9	11
$A(25,2)$	11	13
$A(25,4)$	11	13
$A(25,7)$	11	13

A_Z	a_Z	b_Z
$A(29,3)$	13	15
$A(29,4)$	13	15
$A(29,5)$	13	15
$A(30,3)$	13	16
$A(31,2)$	13	17
$A(31,3)$	13	17
$A(37,3)$	17	19

Notations are the same as in the paper [Grü09] of Grünbaum. From the numbers a_Z and b_Z we can read off the degrees of unexpected curves, $a_Z + 1$.

Examples of unexpected curves in higher degrees

We can also take the configurations $A(n,k)$ and consider only the points with some multiplicity (here multiplicity means the number of lines passing through point). Then we obtain this table:

A	$mult \geq$	#	a_A	b_A
$A^d(13,2)$	4	9	3	5
$A^d(13,2)$	3	13	5	7
$A^d(13,2)$	1	25	11	13
$A^d(17,2)$	4	9	3	5
$A^d(17,4)$	4	9	3	5
$A^d(17,4)$	3	25	11	13
$A^d(19,3)$	4	13	5	7
$A^d(19,3)$	3	25	11	13
$A^d(21,3)$	4	13	5	7
$A^d(21,3)$	$=4$	9	3	5
$A^d(21,3)$	3	37	17	19
$A^d(25,2)$	4	21	9	11

A	$mult \geq$	#	a_A	b_A
$A^d(25,4)$	4	19	7	11
$A^d(25,7)$	4	18	7	10
$A^d(26,3)$	4	19	7	11
$A^d(26,4)$	4	18	7	10
$A^d(27,2)$	4	20	8	11
$A^d(27,3)$	4	21	8	11
$A^d(27,4)$	4	19	7	11
$A^d(28,4)$	4	21	9	11
$A^d(31,3)$	4	31	13	17
$A^d(34,2)$	4	13	5	7
$A^d(37,3)$	4	37	17	19
$A^d(37,3)$	6	13	5	7

1) column "mult" shows what kind of points we consider.
2) column "#" denotes the number of such points.
3) columns "a_A, b_A" shows again the splitting types, so the degree of unexpected curve is $a_A + 1$.

Part 3

A semistability approach

A semistability approach

The following is another approach to the proof of Proposition 2.2. It is shorter and uses the notion of stability of a vector bundle on $\mathbb{P}^n$. We will assume now that $\mathbb{K}$ has characteristic zero and is algebraically closed.

Definition 3.1

We say a vector bundle E on $\mathbb{P}^n$ is **stable** if for every proper subbundle $F \subseteq E$ we have that

$$\frac{c_1(F)}{rank(F)} < \frac{c_1(E)}{rank(E)}$$

where $c_1(F)$ denotes the first Chern class of F. If $<$ is replaced by $\leq$, then we say E is **semistable**.

Stability is related to the existence of unexpected curves as we see now.

A semistability approach

Proposition 3.2

Consider a reduced 0*-dimensional scheme* Z *of* $\mathbb{P}^2$ *and its derivation bundle* $\mathcal{D}_Z$ *with splitting type* (a_Z, b_Z)*. If* Z *admits an unexpected curve, then* $b_Z \geq a_Z + 2$*. In particular,* $\mathcal{D}_Z$ *is not semistable.*

Previous Proposition applies the following Theorem:

Theorem 3.3 (Grauert-Mülich Theorem)

If E *is a semistable rank 2 vector bundle on* $\mathbb{P}^n$*, with splitting type* (a_E, b_E)*, then* $b_E < a_E + 2$*.*

A semistability approach

Lemma 3.4

Let (Z', Z, Z'') be a triple of sets of points of $\mathbb{P}^2$, where $Z = Z' + P$ for some $P \in \mathbb{P}^2$ and Z'' is the projection of Z' to $\mathbb{P}^1$ along P (identifying $\mathbb{P}^1$ with $\mathbb{P}(\mathbb{C}^3/\vec{P})$. Then

(a) *If $|Z|$ is odd, then D_Z is stable if $D_{Z'}$ is stable and $|Z''| > \frac{|Z|+1}{2}$.*

(b) *If $|Z|$ is odd, then D_Z is semistable if $D_{Z'}$ is stable.*

(c) *If $|Z|$ is even, then D_Z is stable if $D_{Z'}$ is semistable and $|Z''| > \frac{d}{2}$.*

(d) *If d is even, then D_Z is stable if $D_{Z'}$ is stable.*

A semistability approach

Proposition 3.5

If Z is a configuration of 9 points over a field of characteristic 0 containing no k-rich lines for $k \geq 4$, then either Z is the 9 point Fermat configuration or it does not admit an unexpected curve.

Sketch of the proof.

1) Show that, for every subconfiguration $W \subseteq Z$ consisting of 5 points, $\mathcal{D}_W$ is stable.
2) Show that, for every subconfiguration $W \subseteq Z$ consisting of 7 points, $\mathcal{D}_W$ is semistable.
3) **Claim:** There is a subset $Z' \subseteq Z$ consisting of 7 points and a point $P \in Z \setminus Z'$ such that at least 4 distinct lines pass through some point of Z' and P, unless Z is the 9 point Fermat configuration.
4) Using the claim and applying Lemma 3.4, prove that $\mathcal{D}_Z$ is semistable unless Z is the 9 point Fermat configuration.

□

A semistability approach

Remarks on Proposition 3.5

- The 9 point Fermat configuration does not admit an unexpected curve by [Coo+16, Theorem 6.11], so Proposition 3.5 is equivalent to Proposition 2.2.
- The proof of Proposition 3.5 could be extended to an arbitrary odd number of points, except that we could have the possibility of a non semistable Fermat configuration as a subconfiguration breaking the proof.

Part 4

A final remark involving de Jonquières transformations

De Jonquières transformations of $\mathbb{P}^2$

Definition 4.1

Let $P, Q_1, \ldots, Q_{2d-2}$ be general points of $\mathbb{P}^2$. The degree d de Jonquières transformation with centers $P, Z = \{Q_1, \ldots, Q_{2d-2}\}$ is the rational map associated to the linear system $\mathscr{L}_{2,d}(d-1, 1^{2d-2})$.

It is well known it is nonspecial of dimension 2 and self-intersection 1, so it induces a birational map $\varphi_{P,Z} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

In order for $\varphi_{P,Z}$ to be birational onto $\mathbb{P}^2$ it is not necessary for $P, Q_1, \ldots, Q_{2d-2}$ to be general.

Lemma 4.2

If Z is a reduced 0-dimensional subscheme of $2d-2$ points of $\mathbb{P}^2$ that does not contain $d+1$ collinear points and if P is general, then $\dim(\mathscr{L}_{2,d,Z}(d-1)) = 2$ *and the associated map is birational.*

Unexpected curves and de Jonquières transformations

So previous Lemma allows us to apply de Jonquières transformations to the problem of unexpected curves.

Corollary 4.3

If $|Z| \leq 2d$ and Z does not contain $d+1$ (or more) collinear points, then $\mathscr{L}_{2,d,Z}(d-1)$ is non special.

Proof

- If $|Z| = 2d-2$, thesis follows by previous Lemma.
- If $|Z| = 2d-1$, fix a subscheme Z' of Z such that $Z = Z' + P'$ for some $P' \in Z$. Then consider the de Jonquières transformation centered at Z' and the fat point P of multiplicity $d-1$. Since it is birational, the elements of $\mathscr{L}_{2,d,Z}(d-1)$ are sent by the associated rational map $\varphi_{P,Z'}$ to lines passing through $\varphi_{P,Z'}(P')$. This means that $\mathscr{L}_{2,d,Z}(d-1)$ is identified with the linear system of lines of $\mathbb{P}^2$ passing through P', of projective dimension 1. Since $\operatorname{expdim}(\mathscr{L}_{2,d,Z}(d-1)) = 1$, we have the thesis.
- A similar argument is valid for $|Z| = 2d$. □

Unexpected curves and de Jonquières transformations

If other words, if Z admits an unexpected curve in degree d, then $|Z| \geq 2d+1$.

In particular, an unexpected curve may appear only if $\mathscr{L}_{2,d,Z}(d-1)$ is expected to be empty. Furthermore, in this case it is unique.

More generally, by using de Jonquières transformations we see that, given a 0-dimensional subscheme Z of at least $2d+1$ distinct points of $\mathbb{P}^2$,

Z admits a degree d irreducible unexpected curve

$\Updownarrow$

$\exists Q = \{Q_1 \ldots, Q_{2d-2}\} \subseteq Z$ such that, called $\varphi_{P,Q}$ the de Jonquières transformation centered at P, Q, the points of $\varphi(Z \setminus Q)$ are collinear.

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