Modelling of Bingham-like fluids with deformable core

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Received 13 September 2005; accepted 27 February 2006

Abstract

We present here two extensions of the usual Bingham model in which we account for deformations in the region where the shear is below the threshold $\tau_o$. In particular, we present two models that consider the material in the above region as a neo-Hookean solid or as a visco-elastic upper convected fluid. Both models are developed in the general 3-D framework of the natural configurations theory. After formulating the basic constitutive equations we focus on two 1-D problems: Shear flow driven by a constant stress and flow in a channel driven by a constant pressure gradient. For both the models, the resulting mathematical problems are free boundary problems involving parabolic and hyperbolic equations.

Keywords: Bingham fluids; Constitutive equations; Dissipation equation; Free boundary problems

1. Introduction

According to the classical model, Bingham fluids, or visco-plastic fluids, are continua which behave as a rigid solid if the applied stress is below a certain threshold (usually called the yield stress and denoted by $\tau_o$) and as a fluid if the stress exceeds that threshold. For the mathematical modelling of Bingham fluids we refer the reader to Bird, Stewart and Lightfoot [1], Rubinstein [2], Duvaut and Lions [3] and to numerous papers on the subject (see, for instance, [4–12]).

Materials which are usually modeled as Bingham fluids are waxy crude oils, i.e. oils characterized by an high content of paraffin and coal water slurries. We refer to [13–15] for the experimental studies on waxy crude oils and to [16–23] for the modelling and the related mathematical literature. For what concerns the rheology of coal–water slurries we refer to [24–27].

The classical Bingham model provides, in some cases, a good schematization of certain materials behavior, but it fails in its basic assumption since we never have a rigid body. In other words, we never have a rigid-fluid transition. For this reason in previous papers [28,29] we modified the usual model considering the material in the solid region no longer rigid, but deformable. Indeed, the practical interest in models which account for the deformability of the material in its non-fluid phase comes from the need to study, both theoretically and experimentally, the effects on the flow caused by the deformations of the “solid”. In particular, waves can develop at the interface between “solid” and liquid and such a fact may produce, for instance, some non negligible effects on the flow.

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doi:10.1016/j.camwa.2006.02.033
In [28,29] we assumed that the body is divided into two domains by a sharp interface whose evolution is not a priori known. In one region the material behaves as a viscous incompressible fluid, whereas in the other like a neo-Hookean solid (see [28]) or like a visco-elastic upper convected Maxwell fluid [30]. The transition is governed by a parameter $a$ that we called the “transition parameter” or “activation parameter”. If $a$ exceeds a certain threshold $a_o$ (related to the yield stress $\tau_o$) the continuum is modeled as a viscous fluid while if $a < a_o$ a visco-elastic [29], or elastic [28], constitutive equation applies.

The models, developed in a general 3-D context, have been deduced exploiting the theory of multiple natural configurations (see Rajagopal et al. [31–35]).

The paper develops as follows: In Section 2 we introduce the basic notations and we derive the constitutive equations for both the elastic and the visco-elastic case. In Section 3 we consider two 1-D applications of the models: shear flow and flow in a channel driven by a constant pressure gradient.

2. The general model

2.1. Notations and basic definitions

Let $K_R$ and $K_t$ be the reference configuration and the actual configuration at time $t$ (the spatial domain occupied by the system at time $t$). The motion of the system is described by the mapping $\chi : (K_R \times \mathbb{R}) \to K_t \subset \mathbb{R}^3$, that associates to the pair $(\vec{p}, t) \in (K_R \times \mathbb{R})$, representing the particle labelled by the vector $\vec{p}$ and the time $t$, one and only one point $\vec{x} \in K_t$, namely

$$\vec{x} = \chi(\vec{p}, t),$$

where $\vec{x}$ represent the Eulerian coordinates. We introduce the deformation gradient\footnote{Grad denotes the gradient operator with respect to coordinates $\vec{p}$, while $\nabla$ denotes the gradient operator with respect to coordinates $\vec{x}$.}

$$\mathbf{F} := \text{Grad} \chi(\vec{p}, t),$$

the velocity gradient\footnote{Recall that it can be shown $\mathbf{L} = \dot{\mathbf{F}} \mathbf{F}^{-1}$. The superposed dot means differentiation along a particle path, i.e. the material derivative $\dot{f} = \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f$.}

$$\mathbf{L} := \nabla \vec{v},$$

and its symmetric part

$$\mathbf{D} := \frac{1}{2} (\mathbf{L} + \mathbf{L}^T).$$

We introduce also the left and right Cauchy–Green stretch tensors

$$\mathbf{B} := \mathbf{F}^T \mathbf{F},$$

$$\mathbf{C} := \mathbf{F}^T \mathbf{F},$$

and assume incompressibility, that is

$$\nabla \cdot \vec{v} = \text{tr} \mathbf{D} = 0.$$
In other words

\[ K_t = K_{1t} \cup K_{2t}, \quad \text{and} \quad K_{1t} \cap K_{2t} = \emptyset \]

where \( K_{it}, i = 1, 2, \) are the current configurations of the viscous and visco-elastic phases, respectively.

Looking at the picture in Fig. 1, we introduce, for each “phase”, the relative natural configuration \( K_{ip}, i = 1, 2 \) (see [31,35]). Roughly speaking this means we are considering a decomposition of the deformation tensor \( F \) in the product

\[ F_1 = F_{1p} G_1, \]
\[ F_2 = F_{2p} G_2, \]

where \( F_i \) is the restriction of \( F \) to the \( i \)th “phase”. Further, since both natural configurations are defined up to rigid rotations (see [35])

\[ F_{ip} = V_{ip} = U_{ip}, \quad i = 1, 2, \]

(1)

where \( V_{ip} \) and \( U_{ip} \) are symmetric positive definite.

Following standard kinematical results on natural configurations theory we introduce

\[ L_{iG} := \dot{G}_i G_i^{-1} \]
\[ D_{iG} := \frac{1}{2}(L_{iG} + L_{iG}^T), \quad i = 1, 2, \]

and

\[ B_{ip} := F_{ip} F_{ip}^T, \]
\[ C_{ip} := F_{ip}^T F_{ip}, \quad i = 1, 2. \]

Since we are dealing with a Bingham-like material, there exists an yield stress \( \tau_o \) to be overcome in order to have purely viscous behavior. We therefore introduce a parameter \( a \) (function of some quantities characterizing the system), that we call the “transition parameter” or “activation parameter” and a threshold value \( a_o \), which in turn will be a function of \( \tau_o \). If \( a \geq a_o \) we model the material as a viscous incompressible fluid, while if \( a < a_o \) as a visco-elastic fluid (in more detail, we model the visco-elastic phase on the basis of the upper convected Maxwell model).

The above means that the transition from the viscous behavior to the visco-elastic behavior is essentially due to a different evolution of the body’s natural configurations. In particular, we stipulate

\[ a := \left( \frac{\eta}{\mu} \right)^2 D_{iG} \cdot D_{iG}, \quad i = 1, 2. \]
and set
\[ a_o := \frac{r_o^2}{2\mu^2}, \]
where \( \eta \) and \( \mu \) are two positive constants representing the viscosity and the elastic modulus, respectively.

2.2. Constitutive assumptions

Following \[36\] we write the dissipation equation (assuming isothermal conditions)
\[ \xi = T \cdot D - \dot{\psi}, \] \( (3) \)
where \( \xi \) is the rate of dissipation, \( \psi \) is the stored energy function, \( T \) is the Cauchy stress and \( T \cdot D \) is the stress power, i.e. the rate at which internal work is done.

We make the assumption that in both phases \( \xi = \xi(D_{IG}, B_{ip}) \) and \( \psi = \psi(B_{ip}) \). Indeed, we do not write constitutive equations for the Cauchy stress \( T \), but we prescribe how the system stores and dissipates energy. Using then the “maximization of rate of dissipation” criterion (see \[33, 35\]) we will get the constitutive equations. We assume that the dissipation \( \xi \) and the stored energy \( \psi \) are
\[ \xi = H(a_o - a) \left[ 2\eta D_{2G} \cdot B_{2p} D_{2G} \right] + H(a - a_o) \left[ 2\eta D_{1G} \cdot D_{1G} \right], \] \( (4) \)
\[ \psi = H(a_o - a) \left[ \frac{\mu}{2} (I \cdot B_{2p} - 3) \right] + H(a - a_o)\psi_o, \] \( (5) \)
where \( \psi_o > 0 \) is a constant and \( H(x) \) is the Heaviside function
\[ H(x) := \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \]

2.2.1. The viscous phase

In the viscous phase \( a \geq a_o \), so that
\[ \xi = 2\eta D_{1G} \cdot D_{1G}, \]
\[ \psi = \psi_o. \]

So, from (3)
\[ T \cdot D = 2\eta D_{1G} \cdot D_{1G}, \]
that, assuming incompressibility (i.e. \( \text{tr}(D) = 0 \)), gives
\[ \begin{cases} T = -PI + 2\eta D, \\ D = D_{1G} \end{cases}, \] \( (6) \)
where \( P \) is the Lagrange multiplier arising from incompressibility. Thus, in the viscous phase, the natural configuration coincides with the actual one, i.e. \( K_{1t} \equiv K_{1p} \).

2.2.2. The visco-elastic phase

In the visco-elastic phase \( a < a_o \), \( H(a_o - a) = 1 \) and \( H(a - a_o) = 0 \). We have
\[ \xi = \xi(D_{2G}, B_{2p}) = 2\eta D_{2G} \cdot B_{2p} D_{2G}, \] \( (7) \)
\[ \psi = \psi (B_{2p}) = \frac{\mu}{2} (I \cdot B_{2p} - 3). \]

\(^3\) \( [\eta] = Pa \cdot s \) and \( [\mu] = Pa. \)
Eq. (3) becomes
\[ \xi (D_{2G}, B_{2p}) = T \cdot D - \frac{\mu}{2} I \cdot \dot{B}_{2p}, \]

It can be proved (see [35]) that
\[ I \cdot \dot{B}_{i\rho} = 2B_{i\rho} \cdot (D - D_{iG}), \quad i = 1, 2, \]
and we get
\[ (T - \mu B_{2p}) \cdot D = \xi (D_{2G}, B_{2p}) - \mu B_{2p} \cdot D_{2G}. \]

Thus
\[ T = -PI + \mu B_{2p}, \]
and
\[ \xi = (T + PI) \cdot D_{2G}, \]
where \( P \) is the Lagrange multiplier due to incompressibility.

To know the evolution of the system we need to determine \( D_{2G} \), i.e. the tensor that governs the evolution of the natural configurations. We use “the maximization of the rate of dissipation” criterion, i.e. we maximize \( \xi \) (given by (7)) under constraint (8) and constraint
\[ I \cdot D_{2G} = 0. \]

The latter has to be fulfilled since the material is incompressible in every configuration and \( K_{2p} \) is one of the possible configurations of the body.

By standard techniques of constrained maximization, we solve
\[ \frac{\partial \xi}{\partial D_{2G}} + \lambda_1 \left( \frac{\partial \xi}{\partial D_{2G}} - T \right) + \lambda_2 I = 0, \]
where \( \lambda_1 \) and \( \lambda_2 \) have to be determined imposing (8) and (9) and where \( B_{2p} \) is kept fixed. After some calculations
\[ 2 \frac{\eta}{\mu} D_{2G} = -\frac{3}{\text{tr}B_{2p}^{-1}} B_{2p}^{-1} + I. \]

Introducing the frame indifferent upper convected time derivative \( \nabla \) (see [30]), we have
\[ \nabla B_{2p} = -2V_{2p} D_{2G} V_{2p}^T, \]
and, by (1),
\[ \nabla B_{2p} = -2F_{2p} D_{2G} F_{2p}^T. \]

So, the Eq. (11) becomes
\[ B_{2p} + \gamma B_{2p} = \frac{3}{\text{tr}B_{2p}^{-1}} I, \]
with
\[ \gamma = \frac{\eta}{\mu}, \quad [\gamma] = s. \]

At this stage the analogy with the upper convected Maxwell model is not clear, since in the latter the deviatoric part of the stress \( S \) evolves according to a constitutive equation of the type
\[ \nabla S + \alpha S = I. \]
However, assuming
\[
B_{2p} = \begin{pmatrix}
1 + B_{11} & B_{12} & B_{13} \\
B_{12} & 1 + B_{22} & B_{23} \\
B_{13} & B_{23} & 1 + B_{33}
\end{pmatrix},
\]
with
\[
|B_{ij}| = O(\varepsilon) \quad i, j = 1, 2, 3, \quad \text{with } \varepsilon \ll 1.
\]
We recover the usual upper convected Maxwell model
\[
\begin{align*}
T &= -PI + \mu B_{2p}, \\
B_{2p} + \gamma \dot{B}_{2p} &= I.
\end{align*}
\]

**Remark 2.1.** When we modify (4), assuming
\[
\xi = H(a - a_o) (2\eta D_{1G} \cdot D_{1G}), \quad \text{(no dissipation for } a < a_o),
\]
from (3)
\[
T \cdot D - \frac{\mu}{2} (I \cdot \dot{B}_{2p}) = 0,
\]
which is equivalent to
\[
(T - \mu B_{2p}) \cdot D - \mu B_{2p} \cdot D_{2G} = 0,
\]
implying
\[
T = -PI + \mu B_{2p} \quad \text{and} \quad D_{2G} = 0.
\]
In this case we have no evolution of the natural configuration. For all time \( t \) we have \( K_{2p} = K_R \), that is \( B_{2p} = B \).
Thus
\[
T = -PI + \mu B,
\]
a neo-Hookean solid.
In such a case we modify the definition of the activation parameter (since \( D_{2G} \) is identically zero). Indeed, we stipulate \( a = |I \cdot B - 3| \).
This is the case in which the region where \( a < a_o \) is modeled as a neo-Hookean elastic solid.

### 3. Application to 1-D cases

In this section we consider two applications of the above described models. We will analyze the following two 1-D cases:

- **Elastic core.** Shear flow of a layer (whose thickness is \( L \)) driven by shear stress \( \sigma \) applied on the top surface (the bottom surface is kept fixed). We assume that the body is separated into two regions by an horizontal interface, whose position with respect to the vertical axis is denoted by \( s(t) \). In particular, we assume that the material is behaves as a viscous fluid for \( s(t) < y \leq L \) and as a neo-Hookean solid for \( 0 \leq y < s(t) \).

- **Visco-elastic core.** Laminar flow in a channel driven by a constant pressure gradient \( \phi_o \). We suppose that the fluid is confined between two parallel planes whose distance is \( 2L \) (due to symmetry reasons we study only the upper part of the channel) and that in \( \{-L < y < -s(t)\} \cup \{s(t) < y < L\} \) the material behaves as a viscous fluid while in \( -s(t) < y < s(t) \) as a visco-elastic fluid (\( y = \pm s(t) \) still denote the two unknown horizontal time-dependent interfaces).

In both the cases the velocity field is directed along the \( x \) axis and is a function of coordinate \( y \) and time \( t \) (see Fig. 2 for more details).
3.1. Elastic core — Shear flow

Considering at first the viscous region, \( s(t) < y < L \), exploiting (6) we write the evolution equation for the velocity field \( v(y, t) \), namely
\[
\rho v_t - \eta v_{yy} = 0,
\]
where \( \rho \) is the material density.

Continuity of the stress on the top surface entails
\[
\eta v_y(L, t) = \sigma,
\]
while the initial condition is denoted by \( v_0(y) \), i.e. \( v(y, 0) = v_0(y) \).

In the solid region, \( 0 < y < s(t) \), we write the evolution equation for the displacement \( f(y, t) \), that, by (15), assumes the form
\[
f_{tt} - c^2 f_{yy} = 0,
\]
where
\[
c = \sqrt{\frac{\mu}{\rho}}
\]
is the speed of sound in the elastic phase.

The non-slip condition on \( y = 0 \), entails \( f(0, t) = 0 \). The initial conditions are \( f(y, 0) = f_0(y) \) and \( f_t(y, 0) = f_1(y) \).

On the interface \( s(t) \) we impose
\[
v(s, t) = f_t(s, t),
\]
ensuring that no “ruptures” occur. At the transition (i.e. when \( a \) tends to \( a_0 \)), the stress tensor becomes
\[
T_o = \begin{pmatrix}
-P + \frac{\tau_o^2}{\mu} & \tau_o & 0 \\
\tau_o & -P & 0 \\
0 & 0 & -P 
\end{pmatrix},
\]
where
\[
\tau_o = \mu \sqrt{a_0} = \mu f_y|_{s(t)},
\]
is the yield stress. So we get
\[
f_y(s(t), t) = \frac{\tau_o}{\mu}.
\]
Assuming then continuity of the normal stress, we have

\[ \eta v_y(s, t) = \mu f_y(s, t). \]

Setting

\[ z(y, t) = v_y(y, t) \quad \text{and} \quad w(y, t) = f_y(y, t), \]

the mathematical problem for \( z \) and \( w \) becomes

\[
\begin{cases}
\rho z_t - \eta z_{yy} = 0, & s(t) < y < L, t > 0 \\
z(s(t), t) = \frac{\tau_o}{\eta}, & t > 0, \\
z(L, t) = \frac{\sigma}{\eta}, & t > 0, \\
z(y, 0) = z_o(y), & s_0 < y < L, \\
z_y(s(t), t) = \frac{\rho}{\eta} \left[ w_y(s, t) - \frac{\tau_o}{\eta} \right] \dot{s} + \frac{\mu}{\eta} w_y(s, t), & t > 0, \\
0 = s_o, & \\
w_{1t} - c^2 w_{yy} = 0, & 0 < y < s(t), t > 0, \\
w(s(t), t) = \frac{\tau_o}{\eta}, & t > 0, \\
w_y(0, t) = 0, & t > 0, \\
w(y, 0) = w_o(y), & 0 < y < s_o, \\
w_{1}(y, 0) = w_1(y), & 0 < y < s_o,
\end{cases}
\tag{16}
\]

where \( s_o, 0 < s_o < L \), is the initial position of the interface and

\[ z_o(y) = v'_o(y), \quad w_o(y) = f'_o(y), \quad w_1(y) = f'_1(y). \]

Problem (16) is a free boundary problem involving a parabolic and an hyperbolic equation whose free boundary \( s(t) \) is governed by a non standard evolution equation.

### 3.2. Visco-elastic core — Flow in a channel

We consider here the case in which the region where the shear stress is below \( \tau_o \) is modeled as a visco-elastic fluid, see (14). We have

\[
\begin{align*}
\tilde{v} &= v(y, t)\tilde{e}_x, & s(t) < y < L, \\
\tilde{v} &= u(y, t)\tilde{e}_x, & 0 < y < s(t),
\end{align*}
\]

where \( v \) and \( u \) represent the velocity fields in the viscous and visco-elastic phase, respectively.

In the viscous phase, from (2) and (6)

\[ a = \frac{1}{2} \left( \frac{\eta v_y}{\mu} \right)^2, \]

so that \( \eta |v_y| \geq \tau_o \). At the interface

\[ |v_y(s(t), t)| = \frac{\tau_o}{\eta}. \tag{17} \]

The dynamics of the fluid in the viscous phase is governed by the following parabolic equation

\[ \rho v_t - \eta v_{yy} = \phi_o, \]

where \( \phi_o \) is the constant pressure gradient driving the flow. The initial condition is \( v(y, 0) = v_o(y) \) and the non-slip condition on the channel wall is \( v(L, t) = 0 \).
Let us now consider the visco-elastic phase. Recalling (13) and (14)

\[
(T)_{ij} = -P \delta_{ij} + \mu B_{ij}, \quad i, j = 1, 2, 3,
\]

where \((-P + \mu)\) has been replaced with \(-P\), the momentum equation yields

\[
\frac{1}{c^2} u_t = \frac{\phi_o}{\mu} + \int_0^t u_{yy}(y, \theta) \exp \left( \frac{\theta - t}{\gamma} \right) d\theta,
\]

where \(\phi_o\) still represents the applied pressure gradient. After multiplying both sides by \(\exp(t/\gamma)\), Eq. (18) can be differentiated with respect to time yielding

\[
\frac{1}{c^2} \left( u_{tt} + \frac{1}{\gamma} u_t \right) - u_{yy}(y, t) = \frac{\phi_o}{\mu \gamma},
\]

which is a dissipative wave or telegraph equation (see [37]).

On \(y = 0\) we assume \(u_y(0, t) = 0\), by symmetry reasons. The initial conditions of (19) are \(u(y, 0) = u_o(y)\) and \(u_t(y, 0) = \frac{c^2 \phi_o}{\mu}\).

**Remark 3.1.** We notice that letting \(\eta \to \infty\) and keeping the elastic modulus \(\mu\) finite (which is equivalent to letting \(\gamma \to \infty\)), Eq. (19) becomes

\[u_{tt} - c^2 u_{yy} = 0.\]

So in this limiting case the region \(a < a_o\) behaves as a neo-Hookean material and we recover the model described in the previous section.

On the free boundary \(y = s(t)\) we still assume no slip, namely

\[u(s(t), t) = v(s(t), t).\]

Further, considering continuity of the normal stress, we get

\[\eta v_y(s(t), t) = \mu \int_0^t u_y(s(t), \theta) \exp \left( \frac{\theta - t}{\gamma} \right) d\theta,\]

that, taking (17) into account, yields

\[\tau_o = \mu \left| \int_0^t u_y(s(t), \theta) \exp \left( \frac{\theta - t}{\gamma} \right) d\theta \right| \tag{20}\]

We remark that, assuming

\[v_o(L) = 0, \quad v'_o(s_o) = -\frac{\tau_o}{\eta} \quad \text{and} \quad v'_o(y) \leq -\frac{\tau_o}{\eta} \quad \forall y \in [s_o, L],\]

the boundary conditions (17) and (20) become

\[v_y(s(t), t) = -\frac{\tau_o}{\eta},\]

and

\[\int_0^t u_y(s(t), \theta) \exp \left( \frac{\theta - t}{\gamma} \right) d\theta = -\frac{\tau_o}{\mu},\]

respectively.

---

4 The choice \(v'_o(s_o) = -\tau_o/\eta\) is reasonable from a physical point of view since the velocity is expected to decrease as \(y\) approaches the wall \(y = L\), where no-slip is imposed.
We are now able to state the mathematical problem

\[
\begin{aligned}
\rho v_t - \eta v_{yy} &= \phi_o, & (y, t) &\in D_1, \\
v(L, t) &= 0, & 0 < t, \\
v(s(t), t) &= u(s(t), t), & 0 < t, \\
v(y, 0) &= v_o(y), & s_o < y < L, \\
v_y(s(t), t) &= -\eta, & 0 < t, \\
\frac{1}{c^2} \left( u_{tt} + \frac{1}{\gamma} u_t \right) - u_{yy} &= \phi_o, & (y, t) &\in D_2, \\
u_y(s(t), t) + \frac{1}{c^2} u_t(s(t), t)\hat{s}(t) &= \frac{1}{\mu} \left[ \phi_o\hat{s}(t) - \frac{\tau_o}{\gamma} \right], & 0 < t, \\
u_y(0, t) &= 0, & 0 < t, \\
u(y, 0) &= u_o(y), & 0 < y < s_o, \\
u_y(0, 0) &= \phi_o, & 0 < y < s_o.
\end{aligned}
\] (21)

Problem (21) is a free boundary problem involving a hyperbolic and a parabolic equation. The peculiar structure of the problem does not allow use of classical tools to prove well posedness. The main difference with the problem studied in [28] is the presence of the dissipative term in the hyperbolic equation (telegraph equation), which prevents a straightforward use of the D’Alembert representation formulas. Existence and uniqueness of a classical solution of problem (21) will be the subject of a forthcoming paper.

Introducing the scaling

\[ v = U\tilde{v}, \quad u = U\tilde{u}, \quad t = TR\tilde{t}, \quad y = L\tilde{y}, \quad s = L\tilde{s}, \]

where

\[ U = \frac{\phi_oL^2}{\eta}, \]

problem (21) can be rewritten as

\[
\begin{aligned}
\chi_1 v_t - v_{yy} &= 1, & s(t) < y < 1, t > 0, \\
v(y, 0) &= v_o(y), & s_o < y < 1, \\
v(1, t) &= 0, & t > 0, \\
v_y(s, t) &= -\chi_2 & i > 0, \\
\chi_3 u_{tt} + \chi_1 u_t - u_{yy} &= 1, & 0 < y < s(t), t > 0, \\
u(s, t) &= v(s, t), & i > 0, \\
u_y(s, t) + \chi_3 \hat{s} u_t(s, t) &= \chi_4 \hat{s} - \chi_2 & i > 0, \\
u(y, 0) &= u_o(y), & 0 < y < s_o, \\
u_y(0, t) &= \chi_5, & 0 < y < s_o, \\
u_y(0, 0) &= \phi_o, & 0 < y < s_o, \\
s(0) &= s_o, & 0 < s_o < 1,
\end{aligned}
\] (22)

where we have omitted the “tildas” and where

\[
\chi_1 = \frac{T_v}{TR}, \quad \chi_2 = \frac{\tau_o}{\phi_o L}, \quad \chi_3 = \left( \frac{T_c}{TR} \right)^2, \quad \chi_4 = \frac{\eta}{\mu TR}, \quad \chi_5 = \frac{\eta TR}{\rho L^2}.
\]

Assuming a characteristic time

\[ TR \gg \sqrt{\frac{\rho L^2}{\mu}}, \]
which is equivalent to $\chi_3 \ll 1$, and assuming $\chi_4 = O(\chi_1)$, problem (22) reduces to

\[
\begin{cases}
    \chi_1 v_t - v_{yy} = 1, & s(t) < y < 1, t > 0, \\
    v(y, 0) = v_o(y), & s_o < y < 1, \\
    v(1, t) = 0, & t > 0, \\
    v_y(s, t) = -\chi_2 & t > 0, \\
    \chi_1 u_t - u_{yy} = 1, & 0 < y < s(t), t > 0, \\
    u(s, t) = v(s, t), & t > 0, \\
    u_y(s, t) = \chi_4 \delta - \chi_2 & t > 0, \\
    u(y, 0) = u_o(y), & 0 < y < s_o, \\
    u_y(0, t) = 0, & t > 0, \\
    s(0) = s_o, & 0 < s_o < 1,
\end{cases}
\] (23)

which is a free boundary problem involving two parabolic equations. Such a problem presents a peculiar characteristic: the datum on the free boundary $s(t)$ is not prescribed and it cannot be included in any of the two-phase free boundary problems classified in [38]. Such a peculiarity prevents the use of the classical techniques (we refer the reader to [39–42] for classical literature). This problem has been studied in [43].

4. Conclusions

We have presented two non-trivial extensions of the classical Bingham model to the case of deformable cores. In particular, we have used the neo-Hookean elastic model and the visco-elastic upper convected Maxwell model to describe the behavior of the domain in which the stress is below the threshold. A key point of the modelling is that the transition from the visco-elastic or elastic behavior to the viscous one is governed by a parameter depending on the natural configurations evolution. We have selected a threshold value (tied to the yield stress of the material) such that the viscous behavior depends on whether the parameter is larger or smaller than such a threshold.

To derive the constitutive equations of the medium we have used a thermodynamical approach, selecting, as constitutive assumptions, the way in which energy is stored and dissipated.

The neo-Hookean elastic model has been applied to a 1-D shear flow problem. The latter has turned out to be a free boundary problem for the heat and the wave equation. Such a problem has been investigated from the mathematical point of view in [28].

The visco-elastic model has been applied to a 1-D flow driven by a pressure gradient. In such a case the mathematical problem generated by the model is a free boundary problem involving a parabolic equation and wave equation with a dissipative term. The mathematical analysis of such a problem has not been fully investigated in its general formulation, even though a simplified case has been studied in [43].

Acknowledgments

The authors would like to express their sincere gratitude to Prof. A. Fasano (Università degli Studi di Firenze) and to Prof. K.R. Rajagopal (Texas A&M University) for their numerous and precious advices. This research was funded by the project P.R.I.N. “Problemi a Frontiera Libera” from the Italian University and Research Ministry (MIUR).

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