Sheaves of quaternionic hyperfunctions and microfunctions

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Sheaves of Quaternionic Hyperfunctions and Microfunctions*

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The sheaf $\mathcal{F}$ of quaternionic hyperfunctions is introduced as the sheaf of boundary values of quaternionic regular functions. A Köthe duality type theorem is established to prove the isomorphism between compactly supported quaternionic hyperfunctions and compactly supported regular functionals. Ordinary differential operators are studied on the sheaf $\mathcal{F}$ with the use of the $C-K$ product. Finally a sheaf of quaternionic microfunctions is introduced as the microlocalization of $\mathcal{F}$, and its main properties are studied.

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INTRODUCTION

In the classical theory of analytic functionals, as envisaged by Fantappié [3], a key result was the algebraic duality theorem which associates to any analytic functional $\mu$ with compact real carrier $K$, a function (the Fantappié indicatrix of $\mu$) holomorphic in $\mathbb{C}\setminus K$. In more modern terms this result was obtained by Köthe [6] in the framework of topological vector spaces, and it was later understood (essentially by Martineau [10]) that the duality theorem actually establishes an isomorphism between the space $B_K(\mathbb{R})$ of real hyperfunctions supported by the compact $K$, and the space $(\mathcal{O}(K))^*$ of analytic functionals carried by $K$. In his book [9], P. Levy proposed the construction of a similar theory, when holomorphic functions are replaced by regular functions of a quaternionic variable. In this paper we show how this is actually possible, and we construct the quaternionic analogue of the sheaf of hyperfunctions; this sheaf is used to prove a version of Köthe's duality theorem for the quaternionic case. We wish to point out, however, that results in this direction had been obtained in [1]. The sheaf of quaternionic hyperfunctions which we construct, can be used to study some properties of differential operators and its singularities can be studied by microlocalization. In the case of Clifford valued monogenic maps,

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results of this kind have been obtained by Sommen in [14, 15]. In particular, after a preliminary section which contains no new results, we introduce and study the sheaf $\mathcal{R}_l$ of left regular functions of a quaternionic variable. The main result in Section 2 is the vanishing of the first cohomology group of an open set in $\mathbb{H}$, with coefficients in the sheaf $\mathcal{R}_l$.

In Section 3, we study boundary values of regular functions and we construct a version of the sheaf of hyperfunctions in this case. Our main result here is Corollary 3.2 in which we prove a natural analogue of the Köthe duality theorem. As a spin-off result we obtain a quaternionic version of the classical Painlevé theorem as well as of the Schwarz reflection principle.

Section 4 is devoted to the study of the $C - K$ product among regular functions, which we employ to study differential operators with regular coefficients, acting on the sheaf of hyperfunctions.

The last section, finally, deals with the microlocalization of quaternionic hyperfunctions and with the study of the sheaf of quaternionic microfunctions. The main properties of this sheaf are studied.

1. PRELIMINARY MATERIAL

In this section we will collect all the basic notions on quaternionic analysis, in order to make this paper as self contained as possible. Our standard references for the notations and the results in this section will be [2, 12, 16], to which we refer the readers for all the proofs.

Let $\mathbb{H}$ denote, as customary, the four dimensional real associative algebra of the quaternions, with the standard basis $\{1,i,j,k\}$ in which $i,j,k$ are imaginary units ($i^2 = j^2 = k^2 = -1$) and $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. We will denote the generic quaternions $q \in \mathbb{H}$ by

$$q = x_0 + ix_1 + jx_2 + kx_3.$$ 

In the sequel $\mathbb{H}$ will be endowed with the euclidean topology of $\mathbb{R}^4$. We define the so called left and right Cauchy-Fueter operators $\partial_l/\partial \overline{q}$, $\partial_r/\partial \overline{q}$ as follows:

$$\frac{\partial_l}{\partial \overline{q}} = \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3},$$

$$\frac{\partial_r}{\partial \overline{q}} = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k.$$ 

**Definition 1.1** Let $U \subseteq \mathbb{H}$ be an open set, and let $f : U \to \mathbb{H}$ be an $\mathbb{R}$-differentiable function. We say that $f$ is left regular if it satisfies the Cauchy-Fueter equation

$$\frac{\partial_l f}{\partial \overline{q}} = 0,$$

and we say that $f$ is right regular if

$$\frac{\partial_r f}{\partial \overline{q}} = 0.$$
We will denote by $\mathcal{R}_l(U)$ and $\mathcal{R}_r(U)$ the $\mathbb{H}$ (vector) spaces of left and right regular functions on $U$.

**Remark 1.1** In the sequel, when no confusion can arise, we will just write $\partial/\partial \bar{q}$ for the operators $\partial_l/\partial \bar{q}$ and $\partial_r/\partial \bar{q}$.

For $U$ an open set in $\mathbb{H}$, let $\mathcal{E}^p(U)$ be the vector space of $C^\infty$ $p$-forms on $U$ with real values. The elements of $\mathcal{E}^p_\mathbb{R}(U) := \mathcal{E}^p(U) \otimes \mathbb{R} \mathbb{H}$ will be called $C^\infty$ $p$-forms with values in $\mathbb{H}$. For our purposes, we introduce a few basic forms:

$$
\begin{align*}
 dq &= dx_0 + idx_1 + jdx_2 + kdx_3, \\
 d\bar{q} &= dx_0 - idx_1 - jdx_2 - kdx_3, \\
 Dq &= dx_1 \wedge dx_2 \wedge dx_3 - idx_0 \wedge dx_2 \wedge dx_3 + jdx_0 \wedge dx_1 \wedge dx_3 \\
 &\quad - kdx_0 \wedge dx_1 \wedge dx_2, \\
 \theta &= dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3.
\end{align*}
$$

Note that $dq, d\bar{q} \in \mathcal{E}^1_\mathbb{H}(U)$, $Dq \in \mathcal{E}^3_\mathbb{H}(U)$, while $\theta \in \mathcal{E}^4_\mathbb{H}(U)$ is the canonical volume form, and the following identity holds:

$$
\overline{dq} \wedge Dq = -Dq \wedge d\bar{q} = 4\theta.
$$

Even though many ideas of quaternionic analysis follow those in complex analysis, there are plenty of striking differences; it is not true, for example, that the product of regular functions is regular, since one can easily show that (for $\partial = \partial_l$)

$$
\frac{\partial(fg)}{\partial q} = \frac{\partial f}{\partial \bar{q}} g + f \frac{\partial g}{\partial x_0} + if \frac{\partial g}{\partial x_1} + jf \frac{\partial g}{\partial x_2} + kf \frac{\partial g}{\partial x_3}.
$$

We will see in Section 4 that a product can indeed be defined, among regular functions, which maintains regularity.

We recall a few basic results in the analysis of regular functions of a quaternionic variable:

**Theorem 1.1 (Cauchy–Fueter I)** Let $f, g : U \subseteq \mathbb{H} \rightarrow \mathbb{H}$ be $C^1$ functions such that $f \in \mathcal{R}_l(U), g \in \mathcal{R}_r(U)$. Then, if $V$ is any open set, $\bar{V} \subset U$, with smooth boundary, we have

$$
\int_{\partial V} g Dqf = 0.
$$

In particular, a $C^1$-function $f : U \rightarrow \mathbb{H}$ is left regular if and only if

$$
\int_{\partial V} Dqf = 0
$$

for any such open set $V$.

**Theorem 1.2 (Cauchy–Fueter II)** Let $f : U \subseteq \mathbb{H} \rightarrow \mathbb{H}$ be a $C^1$, left regular function, and let $V \subseteq U$ be as above. Then for any $q_0 \in V$, and for

$$
G(q) = \frac{q^{-1}}{|q|^2} = \frac{\bar{q}}{|q|^4}
$$

we have
we have

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial V} G(q - q_0)Dqf(q).$$

Remark 1.2 A similar result holds (mutatis mutandis) for right regular functions.

Remark 1.3 The function $G(q)$ is called the Cauchy–Fueter kernel and is both left and right regular.

Theorem 1.3 (Morera) Let $f: U \subseteq \mathbb{H} \rightarrow \mathbb{H}$ be a continuous function such that

$$\int_{\partial V} Dqf = 0$$

for any domain $V$ with smooth boundary, $\bar{V} \subset U$. Then $f \in \mathcal{R}_1(U)$.

It is also possible to provide a power series expansion for regular functions. To this purpose we need some notations: let $m \geq 0$ be an integer and let $\sigma_m$ be the set of all triples $\nu = [m_1, m_2, m_3]$ of nonnegative integers such that $m_1 + m_2 + m_3 = m$; finally, set $e_0 = 1$, $e_1 = i$, $e_2 = j$, $e_3 = k$ and, for $\nu \in \sigma_m$,

$$P_\nu(q) = \frac{1}{m!} \sum (x_0 e_{\lambda_1} - x_{\lambda_1}) \cdots (x_0 e_{\lambda_m} - x_{\lambda_m}),$$

where the summation is extended to all $m$-tuples $(\lambda_1, \ldots, \lambda_m)$ such that $1 \leq \lambda_1, \ldots, \lambda_m \leq 3$ and such that the number of the $\lambda_j$'s equal to $h$ is exactly $m_h$, for $h = 1, 2, 3$. Note that the polynomials $P_\nu$ are homogeneous of degree $m$ on $\mathbb{R}$.

We can now state the following consequence of Theorem 1.2:

Theorem 1.4 Let $f: U \subseteq \mathbb{H} \rightarrow \mathbb{H}$ be a regular function. Let $p \in U$ and let $\delta < \text{dist}(p, \partial U)$. Then, if $|q_0 - p| < \delta$ we have

$$f(q_0) = \sum_{m=0}^{+\infty} \sum_{\nu \in \sigma_m} P_\nu(q_0 - p)a_\nu,$$

where

$$a_\nu = \frac{1}{2\pi^2} \int_{|q-p|=\delta} G_\nu(q-p)Dqf(q),$$

for $G_\nu = \partial^m / \partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}$.

It is a simple computation to show that the $a_\nu$ actually have quite a concrete meaning, namely

$$a_\nu = \text{const} \frac{\partial^m f(p)}{\partial x_1^{m_1} \partial x_2^{m_2} \partial x_3^{m_3}},$$

where the notations are as above.

Remark 1.4 It is unfortunately impossible to obtain a “clean” extension of Theorem 1.2 to the case of several variables, and therefore no power series expansion is known for regular functions on $\mathbb{H}^n$, at least at the moment. It is on the other hand true that any regular function is harmonic, and therefore real analytic, on
\( \mathbb{H}^n = \mathbb{R}^{4n} \); a power series expansion thus exists, but in the \( 4^n \) real variables, not in the \( n \) quaternionic variables. An immediate consequence of the harmonicity of regular functions is the following:

**Theorem 1.5** Let \( U \) be a connected open set in \( \mathbb{H} \) and let \( f, g : U \to \mathbb{H} \) be regular functions which coincide on an open subset of \( U \). Then \( f \equiv g \) on \( U \).

We conclude this short review with the important analogue of the Poincaré lemma for the \( \partial \)-operator:

**Theorem 1.6** Let \( U \subseteq \mathbb{H} \) be open and let \( f \in C^k(U), \ k \geq 1, \ f : U \to \mathbb{H} \). Then there exists \( g \in C^k(U) \) such that \( \partial g / \partial \bar{q} = f \) in \( U \).

## 2. The Sheaf \( \mathcal{R}_l \) of Left Regular Functions

In this section we shall introduce the sheaf of left regular functions on the \( n \)-dimensional quaternionic space \( \mathbb{H}^n \) and its cohomology. For the case \( n = 1 \) we will prove a vanishing of cohomology which is parallel to the classical Mittag-Leffler theorem.

For \( U \) an open set of \( \mathbb{H}^n \), we denote by \( \mathcal{R}_l(U) \) (resp. \( \mathcal{R}_r(U) \)) the right \( \mathbb{H} \) vector space (left \( \mathbb{H} \) vector space) of the left regular functions on \( U \) (right regular functions on \( U \)). Since the theory of \( \mathcal{R}_l(U) \) and of \( \mathcal{R}_r(U) \) are completely parallel, we will only discuss the first case.

**Definition 2.1** We denote by \( \mathcal{R}_l \) the presheaf \( \{ \mathcal{R}_l(U) \}, \ U \subseteq \mathbb{H}^n \), with the usual restriction mappings

\[ \rho_{UV} : \mathcal{R}_l(U) \to \mathcal{R}_l(V), \quad V \subseteq U. \]

**Remark 2.1** It is immediate to see that \( \mathcal{R}_l \) is, indeed, a complete presheaf, and we therefore will denote by \( \mathcal{X}_l \) the sheaf which it defines.

**Remark 2.2** Since, for \( n = 1 \), we have a power series expansion for regular functions, see Theorem 1.4, we can identify the germ \( f_q \in (\mathcal{R}_l)_q \) of a regular function \( f \) at \( q \) with the sequence \( \{ a_\nu \} \) of its coefficients in the power expansion

\[ f(p) = \sum_{m=0}^{+\infty} \sum_{\nu \in \sigma_m} P_\nu (p - q)a_\nu. \]

On the other hand, as we noticed in Remark 1.4, no such an interpretation is available for \( n > 1 \). Also, notice that the stalk \( (\mathcal{R}_l)_q \) of \( \mathcal{R}_l \) at \( q \) is not a ring for the usual product (unlike what happens for the sheaf of germs of holomorphic functions), but only an abelian group under the usual sum. In the sequel, for \( U \) an open set in \( \mathbb{H} \), we will denote by

\[ H^j(U, \mathcal{R}_l) \]

the \( j \)th cohomology group of \( U \) with coefficients in the sheaf \( \mathcal{R}_l \). It is well known that

\[ H^0(U, \mathcal{R}_l) = \Gamma(U, \mathcal{R}_l) = \mathcal{R}_l(U), \]

which is an infinite dimensional right \( \mathbb{H} \)-linear vector space. Therefore the first interesting object to study is the first cohomology group. The following result is a generalization of the well known Mittag-Leffler theorem, [8].
**Theorem 2.1** Let $U$ be any open set in $\mathbb{H}$. Then

$$H^1(U, \mathcal{R}_l) = 0.$$ 

**Proof** Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a locally finite open covering of $U$. We will show that

$$H^1(\mathcal{U}, \mathcal{R}_l) = 0.$$ 

Let therefore $\{g_{ij}\} \in Z^1(\mathcal{U}, \mathcal{R}_l)$, i.e. $g_{ij} \in \mathcal{R}_l(U_i \cap U_j)$ whenever $U_i \cap U_j \neq \emptyset$, and $g_{ij} + g_{ji} = 0$, $g_{ij} + g_{jk} + g_{ki} = 0$, whenever $U_i \cap U_j \cap U_k \neq \emptyset$. Moreover, let $\{\varphi_i\}$ be a partition of unity associated to the covering $\mathcal{U}$, i.e. $\varphi_i \in C^\infty(U_i)$ and $\sum \varphi_i = 1$. Define

$$h_j = \sum_i \varphi_i g_{ij},$$

which is a $C^\infty$ function on $U_j$. Moreover on $U_j \cap U_k$, we have

$$\frac{\partial}{\partial \bar{q}} (h_k - h_j) = \frac{\partial}{\partial \bar{q}} \left( \sum_i \varphi_i g_{ik} - \sum_i \varphi_i g_{ji} \right) = \frac{\partial}{\partial \bar{q}} \left( \sum_i \varphi_i g_{jk} \right) = \frac{\partial g_{jk}}{\partial \bar{q}} = 0.$$

Hence, the function $h = \partial h_j/\partial \bar{q}$ is $C^\infty$ and well defined on all of $U$. By the Poincarè $\bar{\partial}$-lemma for regular functions (Theorem 1.6), we can therefore find $u \in C^\infty(U)$ such that $\partial u/\partial \bar{q} = h$. Finally, if we set

$$g_j = h_j - u$$

on $U_j$, we obtain that

$$\frac{\partial g_j}{\partial \bar{q}} = 0$$

and, on $U_j \cap U_k$,

$$g_k - g_j = h_k - h_j = \sum_i \varphi_i g_{jk} = g_{jk}.$$

This proves that $\{g_{ij}\} \in B^1(\mathcal{U}, \mathcal{R}_l)$, and therefore $H^1(\mathcal{U}, \mathcal{R}_l) = 0$. By taking the inductive limit we conclude the proof. 

**Remark 2.3** When dealing with holomorphic functions, Theorem 2.1 has an obvious interpretation in terms of the first Cousin problem and of meromorphic functions; again, no such an interpretation is possible in this case, since $\mathcal{R}_l(U)$ does not have a ring structure (at least with respect to the classical environment) and therefore no quotient ring can be constructed.

**Remark 2.4** Theorem 2.1 is, in general, false for open sets in $\mathbb{H}^n$, $n > 1$, in view of the Hartogs' phenomenon, [12]. So, if one takes an open set $U \subseteq \mathbb{H}^n$ and a compact $K \subseteq U$, then one can show that, for example, $H^1(U \setminus K; \mathcal{R}_l) \neq 0$. 


3. THE SHEAF $\mathcal{F}$ OF BOUNDARY VALUES OF REGULAR FUNCTIONS

In this section we restrict our attention to one quaternionic variable, and we define a sheaf of “boundary values” of regular functions. We then define a notion of regularity for functions on

$$\{q = (x_0 + ix_1 + jx_2 + kx_3) \in \mathbb{H} : x_0 = 0\} = \mathbb{H},$$

which resembles the familiar notion of real analytic function.

To justify our approach, we begin with a quaternionic version of the familiar Painlevé theorem [11]. We will use the following notations:

$$\mathbb{H}^+ = \{q \in \mathbb{H} : x_0 > 0\},$$

and

$$\mathbb{H}^- = \{q \in \mathbb{H} : x_0 < 0\}.$$

**Theorem 3.1** Let $U$ be an open set in $\mathbb{H}$ and set $U^+ = U \cap \mathbb{H}^+$, $U^- = U \cap \mathbb{H}^-$. Let $F \in \mathcal{R}_c(U^+ \cup U^-)$, $F \in C^0(U)$. Then $F$ is actually regular on $U$.

**Proof** This result is a simple consequence of the quaternionic version of Morera’s theorem (Theorem 1.3). Indeed let $\Gamma$ be an oriented smooth closed three dimensional manifold completely contained in $U$. We want to prove that

$$\int_{\Gamma} DqF = 0. \quad (3.1)$$

Now, if either $\Gamma \subset U^+$ or $\Gamma \subset U^-$, (3.1) is an immediate consequence of the regularity of $F$ in $U^+ \cup U^-$. If, on the other hand, $\Gamma$ is not contained in either $U^+, U^-$, let us call $V = \text{int}(\Gamma)$, with its usual orientation. Then, with the induced orientation (and natural notations),

$$\int_{\Gamma} DqF = \int_{\partial V^+} DqF + \int_{\partial V^-} DqF,$$

since the contributions of the two integrals on $V \cap \tilde{\mathbb{H}}$ cancel each other. But now, since $F \in \mathcal{R}_c(V^+)$, we have that

$$\int_{\Sigma} DqF = 0$$

for any closed smooth three dimensional manifold $\Sigma \subset \subset V^+$. By taking a sequence of such manifolds $\Sigma_n$ converging to $\partial V^+$, and because of the continuity of $F$ on $\bar{U}$, we deduce that

$$\int_{\partial V^+} DqF = 0.$$

A similar argument shows that $\int_{\partial V^-} DqF = 0$, which proves (3.1) and now, by Morera, we conclude. $\blacksquare$
This Theorem can be applied to provide an immediate reflection principle along the ideas of the classical Schwarz theorem: we shall say that a set $V$ in $\mathbf{H}^+$ is strongly symmetric if $(x_0, x_1, x_2, x_3) \in V$ implies that 
$$(x_0, x_1, -x_2, -x_3), (x_0, -x_1, x_2, x_3), (x_0, -x_1, x_2, -x_3)$$
belong to $V$.

**Corollary 3.1** Let $U$ be an open set in $\mathbf{H}$, symmetric with respect to $\mathbf{H}$ and let $U^+ = \{ q \in U : x_0 \geq 0 \}$, $U^- = \{ q \in U : x_0 \leq 0 \}$. Suppose $f = f_0 + if_1 + jf_2 + kf_3$ is a (left)-regular function on the interior of $U^+$, continuous on $U^+$, such that $f$ is real on $\mathbf{H} \cap U$. If $U^+$ is strongly symmetric, then there exists a function $F \in \mathcal{R}_l(U)$, such that $F_{|U^+} = f$.

**Proof** Define, for $q = x_0 + ix_1 + jx_2 + kx_3 \in U^-$
$$\tilde{f}(q) = f_0(-x_0, x_1, x_2, x_3) - if_1(-x_0, x_1, -x_2, -x_3)$$
$$-jf_2(-x_0, -x_1, x_2, -x_3) - kf_3(-x_0, -x_1, -x_2, x_3).$$
It is immediate to see that $\tilde{f}_{|\mathbf{H}} = f_0(0, x_1, x_2, x_3) = f_{|\mathbf{H}}$. Moreover, a direct computation shows that $\partial \tilde{f} / \partial q(q) = 0$, by the Cauchy–Fueter condition. Then
$$F(q) = \begin{cases} f(q) & \text{if } q \in U^+ \\
\tilde{f}(q) & \text{if } q \in U^- \end{cases}$$
satisfies the hypotheses of Theorem 3.1 and is therefore regular on $U$.

It may be interesting to notice that, in the classical Schwarz reflection principle, real and imaginary parts play a symmetrical role; this is no longer true in the quaternionic case, and our Corollary points out the correct generalization.

We can now define, for any open set $U$ in $\mathbf{H}$, the vector space of $\mathbf{H}$-hyperfunctions on $U$:

**Definition 3.1** For $U$ an open set in $\mathbf{H}$, and $V$ an open set in $\mathbf{H}$ such that $U$ is relatively closed in $V$ (e.g. $V$ such that $U = V \cap \mathbf{H}$), we define the space of (left) $\mathbf{H}$-hyperfunctions on $U$ by

$$\mathcal{F}(U) := \frac{\mathcal{R}_l(V \setminus U)}{\mathcal{R}_l(V)}.$$  \hspace{1cm} (3.2)

**Remark 3.1** By Theorem 3.1, the notion of $\mathbf{H}$-hyperfunction represents exactly the “difference of the boundary values” of a function $F \in \mathcal{R}_l(V \setminus U)$ along the “boundary” $\mathbf{H}$. For an elementary treatment of the classical sheaf of hyperfunctions see [7], or the original [13].

**Remark 3.2** We now show that the definition of $\mathcal{F}(U)$ does not depend on the choice of the open set $V$. Indeed let $V_1, V_2$ be two open sets in $\mathbf{H}$ such that $U$ is relatively closed in both of them, and assume $V_1 \subseteq V_2$. Then the restriction mappings
$$\mathcal{R}_l(V_2 \setminus U) \rightarrow \mathcal{R}_l(V_1 \setminus U)$$
and
\[ \mathcal{R}_i(V_2) \to \mathcal{R}_i(V_1) \]
induce a right $\mathcal{H}$ linear map
\[ \rho : \frac{\mathcal{R}_i(V_2 \setminus U)}{\mathcal{R}_i(V_2)} \to \frac{\mathcal{R}_i(V_1 \setminus U)}{\mathcal{R}_i(V_1)} . \]

We note that $\rho$ is an injective map, and we now prove that it is also surjective. Let then $F \in \mathcal{R}_i(V_1 \setminus U)$, and consider the covering
\[ \mathcal{V} = \{ V_1, V_2 \setminus U \} \]
of $V_2$. Now $F$ is defined on $V_1 \cap (V_2 \setminus U) = V_1 \setminus U$ and therefore it defines an element $F \in Z^1(V, \mathcal{R}_i)$. But, by Theorem 2.1, $H_1^i(V, \mathcal{R}_i) = 0$, and therefore $F \in B^1(V, \mathcal{R}_i)$, i.e. there are $G \in \mathcal{R}_i(V_1), N \in \mathcal{R}_i(V_2 \setminus U)$ such that
\[ F = N - G. \]

Therefore $F = \rho(N)$, and so $\rho$ is surjective.

We can now consider the natural presheaf on $\mathcal{H}$, defined by
\[ U \to \mathcal{F}(U), \]
for $U$ any open set in $\mathcal{H}$.

**Theorem 3.2** The presheaf $\{ \mathcal{F}(U) \}$ is, in fact, a flabby sheaf.

**Proof** To verify that $\mathcal{F}$ is a sheaf, let $U$ be an open set of $\mathcal{H}$, and let \{ $U_i$ \} be an open covering of $U$. We must check the following properties:

(i) if $f \in \mathcal{F}(U)$ satisfies $f|_{U_i} = 0$ for all $i$, then $f = 0$ in $\mathcal{F}(U)$;
(ii) if $f_j \in \mathcal{F}(U_j)$ satisfy
\[ f_{j|U_j \cap U_k} = f_{k|U_j \cap U_k}, \]
for all $j, k$ such that $U_j \cap U_k \neq \emptyset$ then there is $f \in \mathcal{F}(U)$ such that $f|_{U_j} = f_j$.

As for (i), it is clearly satisfied, while (ii) needs to be checked. Let therefore $F_j \in \mathcal{R}_i(V_j \setminus U_j)$ be such that $V_j \cap \mathcal{H} = U_j$ and such that $f_j$ is represented by $F_j$. Then, one clearly has that if $V = \cup V_j$, it is $V \cap \mathcal{H} = U$. Define now
\[ G_{ij} = F_j - F_i ; \]
one has, since $f_i = f_j$ on $U_i \cap U_j$, $G_{ij} \in \mathcal{R}_i(V_i \cap V_j)$ and, clearly,
\[ G_{ij} + G_{jk} + G_{ki} = 0. \]
Thus, by Theorem 2.1, there exist $G_i \in \mathcal{R}_i(V_i)$ such that $G_{ij} = G_j - G_i$. We thus conclude that, on $(V_i \cap V_j) \setminus U$,
\[ F_i - G_i = F_j - G_j. \]
We finally define $F \in \mathcal{R}(V \setminus U)$ by setting

$$F(z) = f_I(z) - G(z)$$

on $V \setminus U$. It is immediate to check that $F$ defines, by quotient, an $H$-hyperfunction $f$ for which $f|_I = f_I$. This concludes the proof of the fact that $\mathcal{F}$ is a sheaf on $H$.

To prove the flabbiness of $\mathcal{F}$, let $U' \subseteq U$ be two open sets in $H$, and let $f \in \mathcal{F}(U')$. Take now $V = H \setminus \partial U'$ which is clearly such that $U'$ is relatively closed in $V$ (indeed $U' = V \cap \overline{U'}$). Then $f \in \mathcal{F}(U')$ can be represented by some $F \in \mathcal{R}(V \setminus U')$. Since $V = H \setminus \partial U'$, we have $F \in \mathcal{R}(H \setminus U')$, and since $U' \subseteq U$, $f \in \mathcal{R}(H \setminus U)$. Therefore if we choose $W = H \setminus U'$, we have that $U$ is relatively closed in $W$ and $f \in \mathcal{R}(W \setminus U)$, i.e. $F$ defines an $H$-hyperfunction on $U$, which proves the flabbiness of $\mathcal{F}$.

**Remark 3.3** Since $\mathcal{F}$ is a sheaf, it makes sense to define the support of an $H$-hyperfunction in the usual way. If $K$ is a compact set in $H$, and $f$ is an open set in $H$, we denote by $\mathcal{F}_K(\Omega)$ the space of $H$-hyperfunctions on $\Omega$ whose support is contained in $K$.

**Remark 3.4** It is immediate to check that an $H$-hyperfunction $f \in \mathcal{F}(\Omega)$ has its support contained in $K$ if and only if it has a defining function $F \in \mathcal{R}(V \setminus K)$, for $\Omega$ relatively closed in $V$.

**Definition 3.2** Let $f \in \mathcal{R}_k(\Omega)$, and let $F \in \mathcal{R}(V \setminus K)$ be a defining function for $f$. Then we define the integral of $f$ on $\Omega$ by

$$\int_{\Omega} Df := \int_{\Gamma} DqF$$

where $\Gamma$ is a closed smooth three dimensional manifold, boundary of a four dimensional set $\Sigma$ diffeomorphic to a ball containing $K$, and such that $\Gamma \subseteq V$.

**Remark 3.5** Note that the left hand side of (3.3) has no intrinsic meaning, but is, indeed, defined by (3.3) itself.

**Remark 3.6** Note that the definition of $\int_{\Gamma} DqF$ does not depend on the arbitrary choice of $F$ and of $\Gamma$. Indeed if $F_1$ and $F_2$ are (left)-regular on $V \setminus K$ and define the same $H$-hyperfunction $f$, then

$$\int_{\Gamma} DqF_1 = \int_{\Gamma} DqF_2 = \int_{\Gamma} Dq(F_1 - F_2);$$

but $F_1 - F_2 \in \mathcal{R}(V)$, and therefore, by Theorem 1.1, we get

$$\int_{\Gamma} DqF_1 = \int_{\Gamma} DqF_2.$$
now we note that $F$ is regular on $\Sigma_1 \setminus \Sigma_2$ and therefore, by Cauchy–Fueter’s Theorem,
\[ \int_{\partial(\Sigma_1 \setminus \Sigma_2)} DqF = 0. \]
This proves that Definition 3.2 is well posed.

**Remark 3.7** It is immediate to note that the Cauchy–Fueter kernel defines an $H$-hyperfunction $[G(q - q_0)]$ which is clearly the $H$-version of the Dirac delta, as it is obvious from the definition of the integral of compactly supported hyperfunctions.

In the classical hyperfunction theory, it is a basic fact that the space $B_K(\mathbb{R})$ can be naturally identified with the space of analytic functionals supported by $K$. It is now possible to prove a similar result for the case of $H$-hyperfunctions.

**DEFINITION 3.3** Let $K$ be a compact set in $H$. We define the space of germs of right $H$-analytic functions on $K$ by setting
\[ \mathcal{G}(K) = \text{ind. lim. } \mathcal{R}_f(U), \quad (3.4) \]
and endowing $\mathcal{G}(K)$ with its natural inductive limit topology.

The classical Fantappiè–Köthe–Martineau duality theorem [6] can now be generalized as follows:

**THEOREM 3.3** Let $K$ be a compact set in $H$ and let $V$ be an open set containing $K$. Then
\[ (\mathcal{G}(K))' \cong \mathcal{R}_f(V \setminus K)/\mathcal{R}_f(V), \]
where $(\mathcal{G}(K))'$ denotes the space of left $H$ linear continuous functionals on $\mathcal{G}(K)$.

**Proof** Let $f \in \mathcal{R}_f(V \setminus K)/\mathcal{R}_f(V)$ be represented by some function $F \in \mathcal{R}_f(V \setminus K)$. A functional $\mu_f$ on $\mathcal{G}(K)$ can be now defined as follows: for any $g \in \mathcal{G}(K)$, denote again by $g$ a right regular extension of its to some neighbourhood $U$ of $K$, $U \subseteq V$ (such an extension always exists by (3.4)), and define
\[ \langle \mu_f, g \rangle = \int_{\partial U} g DqF. \quad (3.5) \]
Note that (3.5) is a well posed definition:

(i) $\langle \mu_f, g \rangle$ does not depend on $U$; indeed if $U$ is replaced by an analogous $U'$, say $U \subseteq U'$, we have
\[ \int_{\partial U'} g DqF - \int_{\partial U} g DqF = \int_{\partial(U' \setminus U)} g DqF = 0, \]
where the last equality is again a consequence of Cauchy–Fueter’s theorem, since $g$ is right regular on $U' \setminus U$ and $F$ is left regular in $U' \setminus U$;

(ii) $\langle \mu_f, g \rangle$ does not depend on the extension of $g$, since all the extensions coincide, in suitably small neighbourhoods of $K$;
(iii) \((\mu|\mathcal{H})\) does not depend on the choice of \(F\) (again by Cauchy–Fueter).

It is obvious to show that \(\mu_f\) is left \(\mathcal{H}\)-linear, and that \(\mu_f\) is continuous on \(G(K)\).

We denote by \(T\) the map \(T : R_l(V) / R_l(V) \rightarrow (G(K))'\) which associates to \(f\) the functional \(\mu_f\); note that this map is right \(\mathcal{H}\)-linear. To conclude the proof it is sufficient to construct a map \(S : (G(K))' \rightarrow R_l(V) / R_l(V)\) which inverts \(T\).

Let therefore \(\mu \in (G(K))'\), and define a new functional

\[
F(q) := (\mu_q, G(p - q)),
\]

where \(p, q \in \mathcal{H}, G(p - q)\) is the Cauchy–Fueter kernel, and \(\mu_q\) indicates that \(\mu\) acts on \(G(p - q)\) as a functional in \(\mu\) (\(q\) being regarded as a parameter). In view of \([5]\), it is fair to call \(F(q)\) the Fantappé indicatrix of \(\mu\). It is immediate to note that

\[
F \in R_l(V) / R_l(V);
\]

indeed, for \(p \in K\),

\[
\frac{\partial}{\partial q} F = \frac{\partial}{\partial q} (\mu_q, G(p - q)) = \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} + j \frac{\partial}{\partial x_3} + k \frac{\partial}{\partial x_4} \right) (\mu_q, G(p - q)) = \left( \mu_q, \frac{\partial}{\partial q} G(p - q) \right),
\]

since \(\mu\) is left \(\mathcal{H}\) linear. Finally \((\partial / \partial q) G(p - q) = 0\) when \(q \notin K\), and \(F \in R_l(V) / R_l(V)\).

Now, \(F\) defines immediately an object \(f\) in \(R_l(V) / R_l(V)\); we thus have a map

\[
S : (G(K))' \rightarrow R_l(V) / R_l(V);
\]

whio maps \(\mu\) to \(f = S(\mu)\). Note that both \(S\) and \(T\) are right \(\mathcal{H}\)-linear maps, and we will now prove that \(f \circ S = \text{id}, S \circ T = \text{id}\).

Indeed, let \(\mu \in (G(K))'\). Then \(S(\mu)\) is the left regular function

\[
F(q) = (\mu, G(p - q)) \in R_l(V) / R_l(V);
\]

now \(T(S(\mu))\) is the functional \(\mu_f\), which, on any right \(\mathcal{H}\) analytic function \(g\), acts as follows (keeping in mind the left-linearity of \(\mu\) and the right regularity of \(g\)):

\[
\langle \mu_f, g \rangle = \int_{\mathcal{H}} g dq F(q) = \int_{\mathcal{H}} g dq (\mu, G(p - q)) = \left( \mu, \int_{\mathcal{H}} g dq G(p - q) \right) = (\mu, g).
\]

Therefore \(T(S(\mu)) = \mu\).
Conversely, if \( F \in \mathcal{R}_r(V \setminus K) \), we have that \( T(F) \) is the functional \( \mu_F \) which, on each right regular \( \mathbb{H} \)-analytic function \( g \), acts as follows:

\[
\langle \mu_F, g \rangle = \int_{\partial U} g \, DqF.
\]

Finally

\[
S(\mu_F) = S(T(F))(q) = \langle \mu_F, G(p - q) \rangle = \int_{\partial U} G(p - q) \, DqF = F,
\]

by the left regularity of \( F \).

\[\text{COROLLARY 3.2} \quad \text{If } K \text{ is a compact set in } \mathbb{H}, \text{ then}
\]

\[
(G(K))' \cong \mathcal{F}_K(\mathbb{H}).
\]

Remark 3.8 Theorem 3.3 and Corollary 3.2 are a natural generalization of the classical treatment of analytic functionals, to the case of regular functions of a quaternion variable.

4. ORDINARY DIFFERENTIAL OPERATORS

In this section, we wish to demonstrate the use of the sheaf \( \mathcal{F} \) as far as the study of ordinary differential operators is concerned. We first need to recall an important notion which we have briefly mentioned in Section 2. As we know, the product of two regular functions need not be regular, because of the non-commutativity of the algebra \( \mathbb{H} \). It is however possible to define a different notion of product which makes \( \mathcal{R} \) into an algebra. To this purpose, we provide a brief sketch of how to construct the so called \( C - K \) product (Cauchy–Kowalewski product), and we refer the reader to [2, 4, 5] for more details. As customary, for \( x \) a point in \( \mathbb{H} \), and \( f \) a regular function in an open set \( U \) which contains \( x \), we denote by \( f(x + 0) \) and by \( f(x - 0) \), respectively, the boundary values of \( f \) at \( x \), from above and from below respectively. The following result can be proved [12]:

\[\text{THEOREM 4.1} \quad \text{Let } S \text{ be an analytic hypersurface of } \mathbb{H} \text{ and let } f : S \to \mathbb{H} \text{ be an analytic function. Then there exists a unique left regular function } F, \text{ defined in a neighbourhood of } S, \text{ such that } F|_S = f.
\]

\[\text{DEFINITION 4.1} \quad \text{Let } V \text{ be an open set in } \mathbb{H}, \text{ and let } f \text{ be an analytic function in } V. \text{ The regular extension } F \text{ of } f \text{ we described before is called the left Cauchy–Kowalewski extension (} C - K \text{ extension) of } f, \text{ and is denoted by } f^*.
\]

The above definition makes sense in view of Theorem 1.5. A final important remark on \( C - K \) extensions concerning the commutativity of the operations of \( C - K \) extension and of differentiation is the following: in symbols

\[
\partial^\beta (f^*) = \{\partial^\beta f\}^*
\]

with the usual notations.
We now consider \( f \) and \( g \) to be left regular functions on some common open set \( V \) which has a non-empty intersection with \( \mathbb{H} \). Then their restrictions to \( \mathbb{H} \) are \( \mathbb{H} \)-valued analytic functions in \( V \cap \mathbb{H} \), and therefore their product is, too, analytic. Such a product admits, in view of our previous remarks, a left \( C-K \) extension; this extension is denoted by 
\[
    f \ast g
\]
and is called the Cauchy-Kowalewski product (\( C-K \) product) of \( f \) and \( g \).

We can now discuss linear ordinary differential equations. To begin with, let us note that a few standard operations can be defined on \( \mathbb{H} \)-hyperfunctions, by just resorting to a representative regular function. So, if we write \( f = [F] \), we mean that \( f \) is a \( \mathbb{H} \)-hyperfunction and that \( F \) is a left regular function which represents \( f \) in the quotient (3.2). It is then obvious that \( \mathbb{H} \)-hyperfunctions can be added by just adding the corresponding representatives. Of more interest is the fact that they can be differentiated in an analogous way. To describe how this can be actually done, let us go back for a second to the \( C-K \) extension and to the \( C-K \) product. It is not difficult to see that if \( x_i \) (\( i = 1, 2, 3 \)) are the real variables in \( \mathbb{H} \), then their \( C-K \) extensions are given by
\[
    z = x_i e_i - x_i e_i, \quad i = 1, 2, 3.
\]

We can use this fact to extend differential operators from the space \( \mathbb{H} \) to all of \( \mathbb{H} \). Indeed, the operator \( \partial/\partial x_i \) extends in a natural way to the operator \( \partial/\partial z_i \), defined by
\[
    \partial/\partial z_i = (\partial/\partial x_1) \delta_{i1} - (\partial/\partial x_2) \delta_{i2} - (\partial/\partial x_3) \delta_{i3}.
\]

On the other hand, every analytic function \( f \) on an open set in \( \mathbb{H} \) extends (uniquely) to its \( C-K \) extension which we have denoted by \( F \). We can therefore consider linear ordinary differential operators acting on the sheaf of \( \mathbb{H} \)-hyperfunctions as follows:

for \( f = [F] \) an \( \mathbb{H} \)-hyperfunction on an open set \( V \), and for
\[
    P = P(\partial d/\partial x_i) = \sum_{i=1}^{3} a_i d^{i}/dz^{i}
\]
where \( d^{i}/dz = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3) \), one defines
\[
    P(d^{i}/dz)f = [P(d^{i}/dz)F]
\]
where now \( z = (z_1, z_2, z_3) \), and \( d^{i}/dz = (\partial/\partial z_1, \partial/\partial z_2, \partial/\partial z_3) \). Note that if \( F \) is regular, also \( P(d^{i}/dz)F \) is regular in view of the definition of \( d^{i}/dz \). This enables us to deal with differential equations on \( \mathbb{H} \)-hyperfunctions.

We now want to use this definition to show how things become fairly simple in this realm.

As far as we are aware of, no general existence results are available for partial differential equations in the space of regular functions. In our next result we use the analyticity of regular functions to prove a global existence theorem on relatively
compact open sets. The main difficulty, here, is that the global theory of analytic coefficients partial differential equations is still under deep investigation. In a series of papers [17–19], Kawai and Takei have shown that if the relatively compact open set \( \Omega \) in \( \mathbb{R}^n \) and the differential operator \( P(x,d/dx) \) are compatible (they use the terminology of strong \( P \)-convexity, and we refer the reader to [18] for details and examples) then \( P(x,d/dx) : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega) \) is surjective.

This allows us to prove the following result:

**Theorem 4.2** Let \( V \) be a relatively compact open set in \( \mathbb{H} \), let \( F \) be regular in \( V \) and suppose

\[
P(d/dz)
\]

be a partial differential operator such that \( a_0 \equiv 0 \), and \( V \) is \( P \)-convex with respect to the operator \( P \) considered as an operator in \( \mathbb{R}^4 \). Then there exists a regular solution \( G \) to the equation

\[
P(d/dz)G = F. \tag{4.2}
\]

**Proof** From Kawai's result [18] we deduce the existence of a real analytic solution \( H \) to the equation

\[
P(d/dz)H = F.
\]

In general, \( H \) will not be regular (i.e. \( \partial H/\partial q \) may not be identically zero) and therefore we now look for a function \( K \) such that

\[
\begin{align*}
\frac{\partial K}{\partial q} &= -\frac{\partial H}{\partial q} \\
P(d/dz)K &= 0
\end{align*}
\]

so that the function

\[
G := H + K
\]

is a regular solution of (4.2). We now note that (4.3) can be rewritten as:

\[
\begin{align*}
\frac{\partial K}{\partial z_1} &= 0 \\
\frac{\partial K}{\partial z_2} &= 0 \\
\frac{\partial K}{\partial z_3} &= 0 \\
\frac{\partial K}{\partial q} &= -\frac{\partial H}{\partial q}
\end{align*}
\]

(4.4)
for $z_1, z_2, z_3$ as in (4.1). Keeping in mind (4.1) and the definition of $\partial / \partial \overline{q}$, (4.4) becomes

\[
\begin{align*}
\frac{\partial K}{\partial x_1} &= \frac{\partial K}{\partial x_0}i, \\
\frac{\partial K}{\partial x_2} &= \frac{\partial K}{\partial x_0}j, \\
\frac{\partial K}{\partial x_3} &= \frac{\partial K}{\partial x_0}k, \\
\frac{\partial K}{\partial x_0} + i \frac{\partial K}{\partial x_1} + j \frac{\partial K}{\partial x_2} + k \frac{\partial K}{\partial x_3} &= - \frac{\partial H}{\partial \overline{q}}.
\end{align*}
\]

Finally (4.5) is rewritten as

\[
\begin{align*}
K_{x_1} &= K_{x_0}i, \\
K_{x_2} &= K_{x_0}j, \\
K_{x_3} &= K_{x_0}k, \\
-2K_{x_0} &= - \frac{\partial H}{\partial \overline{q}}.
\end{align*}
\]

This system is now immediately solved by integration, and the theorem is proved. ■

We can now prove two results which should be sufficient to illustrate the interest of our approach.

**Theorem 4.3** Let $V$ be a relatively compact open set in $\mathbb{H}$ which admits a neighborhood $U$ in $\mathbb{H}$ such that $U \setminus V$ is $P$-convex. Then for any $f$ in $\mathcal{F}(V)$, there exists $g$ in $\mathcal{F}(V)$ such that

\[
P(d/dx)g = f.
\]

**Proof** Let us consider a neighborhood $U$ of $V$, to which all the coefficients of the differential polynomial $P$ can be $C-K$ extended, and choose $U$ so that $U \setminus V$ is $P$-convex as in [18]. Let then $F$ be a defining regular function for $f$ in $U \setminus V$. By Theorem 4.2, it is now possible to solve, in $\mathcal{F}(U \setminus V)$, the equation

\[
P(d/dz)G = F.
\]

But then, the required solution is exactly $g = [G]$, which concludes the proof. ■

**Theorem 4.4** Let $P, V$ be as above and let $W$ be an open subset of the open set $V$, in $\tilde{\mathbb{H}}$. If $f$ belongs to $\mathcal{F}(V)$, then any solution $h$ in $\mathcal{F}(W)$ to equation (4.6), can be extended to a solution $g$ in $\mathcal{F}(V)$ to the same equation.

Note that the statement is clearly false if we replace the sheaf $\mathcal{F}$ by the sheaf $\mathcal{G}$ of analytic functions.

**Proof** This result is actually an almost immediate consequence of the previous surjectivity result, and of the Mittag–Leffler Theorem (Theorem 2.1). Indeed, by Theorem 4.3, it is sufficient to consider the homogeneous equation

\[
P(d/dx)g(x) = 0.
\]
Let $U$ be the neighbourhood of $V$ constructed in the proof of Theorem 4.3. Then $U$ can be modified to provide a neighbourhood of $W$, which we denote by $Y$. Let $H$ be the defining function of the solution $h$ of (4.7), which, of course, is left regular on $Y \setminus W = U \setminus V$ (this equality can always be obtained, through a suitable choice of $Y$). One then notes that $P(d/dz)H(z)$ is left regular on $Y$ (by the properties of the $C \times K$ product), and therefore the proof of the Theorem will be complete if we will prove that there is a solution $N(z)$, left regular on $Y$, such that the difference

$$ P(d/dz)H(z) - P(d/dz)N(z) $$

can be continued to a left regular function on $U$. If this is possible, then the requested solution will be

$$ g = [H - N]. $$

To prove the existence of $N$, we cover $U$ with two open sets $U_1$ and $U_2$, such that $U_2$ does not intersect $V \setminus W$, and $U_1 \setminus (V \setminus W)$ is $P$-convex. One can then find a regular solution $N_0$ to the equation

$$ P(d/dz)N_0 = P(d/dz)H $$

in $U_1 \setminus (V \setminus W)$. But now we can apply Mittag–Leffler Theorem to obtain the existence of left regular functions $N_1$ and $N_2$ on $U_1$ and $U_2$ respectively, such that, on the intersection of the two open sets, it is

$$ N_0 = N_2 - N_1. $$

The Theorem now follows immediately.

One may even wish to push further to consider infinite order differential operators (with constant coefficients). Let us recall that the homogeneous polynomials

$$ P_\nu(q) = \frac{1}{m!} \sum (x_0 e_{\lambda_1} - x_{\lambda_1}) \cdots (x_0 e_{\lambda_m} - x_{\lambda_m}) $$

(see Section 1 for the meaning of the notations) appear in the series expansion of a regular function. It is natural to consider the associated differential operator

$$ P_\nu \left( \frac{d}{dz} \right) := \frac{1}{m!} \sum \frac{\partial}{\partial z_{\lambda_1}} \cdots \frac{\partial}{\partial z_{\lambda_m}}; $$

since

$$ \frac{\partial}{\partial q} \left( \frac{\partial}{\partial z_i} \right) = \frac{\partial}{\partial z_i} \left( \frac{\partial}{\partial q} \right), $$

one has that

$$ P_\nu \left( \frac{d}{dz} \right) \frac{\partial}{\partial q} = \frac{\partial}{\partial q} P_\nu \left( \frac{d}{dz} \right), $$

and therefore

$$ P_\nu(D) : \mathbb{R} \to \mathbb{R} $$

is a sheaf homomorphism.
It is a natural question to ask which growth condition one should impose on constant coefficients \( \{a_r\} \) so that
\[
\sum_{n=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n} P_{\alpha}(D)a_{\alpha}
\]
is a sheaf homomorphism on \( R \).

**Theorem 4.5**

If
\[
\lim_{|\alpha| \to +\infty} \sqrt{\alpha!}|a_{\alpha}| = 0,
\]
then (4.9) defines a sheaf homomorphism from \( R \) to \( R \) and, therefore, also a sheaf homomorphism from \( F \) to \( F \). Such a homomorphism will be called infinite order differential operator.

**Proof**

Let \( f \) be a germ of regular function in a ball \( \Delta = \{ q : |q - q_0| < \delta \} \), and let \( |f(q)| \leq M \) on \( \Delta \). By the Cauchy's inequality (immediate consequence of Theorem 1.4), one has
\[
\left| \frac{\partial^n f(q)}{x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}} \right| \leq \frac{K! M}{\alpha^n}
\]
on \( \Delta \), and therefore, by noticing that for \( t = 1, 2, 3 \), each \( \partial/\partial q_t \) is a linear combination of \( \partial/\partial x_1 \) and \( \partial/\partial x_2 \), we deduce
\[
\left| \frac{\partial^n f(q)}{\partial x_1^{q_1} \partial x_2^{q_2} \cdots \partial x_n^{q_n}} \right| \leq \frac{(K + 3)K! M}{\delta^k}
\]
on \( \Delta \).

From (4.10) we know that for every \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that
\[
|a_{\alpha}| |\alpha|^\epsilon \leq C_\epsilon |\alpha|^\epsilon,
\]
and therefore, for \( \epsilon = \delta/2 \), we get
\[
\left| \frac{\partial^n f(q)}{\partial x_1^{q_1} \partial x_2^{q_2} \cdots \partial x_n^{q_n}} \right| \leq C_\epsilon \frac{M(K + 3)}{\delta^k},
\]
which is enough to conclude that the series \( \sum_{n=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n} P_{\alpha}(d/dz)f_{\alpha} \) converges uniformly in \( |q - q_0| \leq \delta \), and therefore determines a regular function.

**5. MICROLOCALIZATION**

We now want to proceed to the study of the singularities of \( \mathbb{H} \)-hyperfunctions. We first need to establish some simple facts on hyperfunctions (let us emphasize that we only consider one-dimensional objects). Let us begin by giving some concrete examples of \( \mathbb{H} \)-hyperfunctions.

Let \( F \in \mathbb{R}(\mathbb{H}^+) \). Then the function
\[
\tilde{F}^+ = \begin{cases} F & \text{on } \mathbb{H}^+ \\ 0 & \text{on } \mathbb{H}^- \end{cases}
\]
clearly belongs to $\mathcal{R}(\mathbb{H}\backslash\mathbb{H})$ and it therefore defines an element in $\mathcal{F}(\mathbb{H})$ which represents the boundary value of $F$. By using a notation which is classical in the theory of hyperfunctions, we may write

$$b_+ F = [\hat{F}^+] = F(ix_1 + jx_2 + kx_3 + 0).$$

In an analogous way one can define

$$b_- F = F(ix_1 + jx_2 + kx_3 - 0) = [\hat{F}^-],$$

for $F \in \mathcal{R}(\mathbb{H}^-)$ and

$$\hat{F}^- = \begin{cases} 0 & \text{on } \mathbb{H}^+ \\ F & \text{on } \mathbb{H}^- \end{cases}.$$ 

It is then clear that if $F \in \mathcal{R}(\mathbb{H}\backslash\mathbb{H})$, then (with obvious meaning of symbols)

$$[F] = b_+ F + b_- F.$$

Also, if $f \in \mathcal{G}_s(\mathbb{H})$, or $f \in \mathcal{G}_s(U)$, for $U$ any open set in $\mathbb{H}$, we see that $f$ defines in a natural way an $\mathbb{H}$-hyperfunction on $\mathbb{H}$, or on $U$, which is obtained by considering a regular extension $\hat{f}$ of $f$ to a suitable neighbourhood of $U$ in $\mathbb{H}$, and then the $\mathbb{H}$-hyperfunction $[\hat{f}]$ defined by $\hat{f}$ itself.

We therefore have a natural sheaf injection $i$

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{F}$$

where the sheaves are defined on the topological space $\mathbb{H}$.

In order to study the singularities of the elements of the sheaf $\mathcal{F}$, one introduces, on $\mathbb{H}$, the sheaf $\mathcal{F}/i\mathcal{G}$. As it is well known, $\mathcal{F}/i\mathcal{G}$ need not be a sheaf (quotient of sheaves need not be sheaves) but, in this particular case, it can be shown that $\mathcal{F}/i\mathcal{G}$ is indeed a sheaf. Let us now consider $S^*\mathbb{H}$ to be the topological space $(\mathbb{H}, +\infty) \cup (\mathbb{H}, -\infty)$ made of two copies of $\mathbb{H}$, with the topology being the one induced by the basis $(I, +\infty) \cup (J, -\infty)$ for $I, J$ open sets in $\mathbb{H}$.

We wish to define a sheaf of $\mathbb{H}$-microfunctions on $S^*\mathbb{H}$. To this purpose we give the following definition:

**Definition 5.1** Let $x_0 \in \mathbb{H}$. We say that a hyperfunction $f$ defined in a neighbourhood of $x_0$ is microregular in $(x_0 + \infty)$ if, in a (possibly different) neighbourhood of $x_0$, $f$ is the boundary value from below of a regular function $F$.

**Remark 5.1** A similar (and symmetrical) definition can be given for the notion of microregularity in $(x_0 - \infty)$. Also note that a hyperfunction is the image of an analytic function if and only if it is both microregular in $(x_0 + \infty)$ and in $(x_0 - \infty)$.

**Definition 5.2** Let $f \in \mathcal{F}$. We define its singularity support $s.s.(f)$ to be the set of points of $S^*\mathbb{H}$ in which $f$ is not microregular.

**Remark 5.2** It is immediate to note that if $f \in \mathcal{F}$ and $P(\partial/\partial x)$ is an infinite order differential operator, then

$$s.s.\left( P\left( \frac{\partial}{\partial x} \right) f \right) \subseteq s.s.(f).$$
DEFINITION 5.3 For any open set \( U \subseteq S^*\hat{H} \) we define

\[
\mathcal{C}(U) = \frac{\mathcal{F}(\hat{H})}{\{ f \in \mathcal{F}(\hat{H}) : s.s.(f) \cap U = \emptyset \}}.
\]

It is now possible to show that the data \( \{U, \mathcal{C}(U)\} \) for all open sets \( U \) in \( S^*\hat{H} \), give a sheaf on \( S^*\hat{H} \), which we can call the sheaf of \( \hat{H} \)-microfunctions. Our main results in this area are as follows:

(a) \( \mathcal{C} \) is a flabby sheaf on \( S^*\hat{H} \);
(b) the following short sequence of sheaves on \( \hat{H} \) is exact:

\[
0 \to \mathcal{G} \to \mathcal{F} \to \pi_*\mathcal{C} \to 0,
\]

where \( \pi : S^*\hat{H} \to \hat{H} \) is the canonical projection of \( S^*\hat{H} \) onto \( \hat{H} \).

To prove these two statements, let us begin by looking at the sheaf inclusion

\[
\mathcal{G} \to \mathcal{F},
\]

and by considering the quotient sheaf \( \mathcal{F}/\mathcal{G} \) which can be considered as the sheaf which describes the singularities of quaternionic hyperfunctions.

THEOREM 5.1 The sheaf \( \mathcal{F}/\mathcal{G} \) is a flabby sheaf, and, for any open set \( U \subseteq \hat{H} \),

\[
\frac{\mathcal{F}}{\mathcal{G}}(U) = \frac{\mathcal{F}(U)}{\mathcal{G}(U)}.
\]

Finally for every \( f \in \mathcal{F}(U) \), s.s.(\( f \)) equals the support of the equivalence class of \( f \) in \( \mathcal{F}(U)/\mathcal{G}(U) \), regarded as an element of \( \mathcal{F}/\mathcal{G}(U) \).

Proof The flabbiness of \( \mathcal{F}/\mathcal{G} \) is an immediate consequence of the flabbiness of \( \mathcal{F} \) and of (5.2).

By the definition of quotient sheaf, we know that the natural mapping

\[
\rho : \frac{\mathcal{F}(U)}{\mathcal{G}(U)} \to \frac{\mathcal{F}}{\mathcal{G}}(U)
\]

is injective (this is because, in the quotient presheaf, local vanishing implies global vanishing). It is therefore sufficient to prove that \( \rho \) is surjective. Let \( [f] \in \mathcal{F}/\mathcal{G}(U) \); we know that, in the neighbourhood of each \( x \in U \), \([f]\) can be represented by a section of \( \mathcal{F} \), so that there exists an open covering \( \{U_i\} \) of \( U \), and \( f_j \in \mathcal{F}(U) \) such that

\[
[f]_{|U_j} = f_j \mod \mathcal{G}.
\]

If we now set

\[
\tilde{g}_{ij} := f_{j_{|U_j \cap U_i}} - f_{i_{|U_j \cap U_i}},
\]

we have that \( \{\tilde{g}_{ij}\} \in Z^1(U,G) \). By regular continuation of the \( g_{ij} \) to \( \tilde{g}_{ij} \) regular in suitable neighbourhoods of the \( U_j \) in \( H \), we obtain

\[
\{\tilde{g}_{ij}\} \in Z^1(U,R).
\]
But we can now apply Mittag-Leffler theorem to conclude the existence of $g_i \in G(U_i)$ such that

$$g_{ij} = g_j - g_i \quad \text{on} \quad U_i \cap U_j.$$ 

This implies that the family

$$\{f_i - g_i\}$$

can be glued together to create an element in $F(U)$. The theorem is proved. ■

Remark 5.3 Note that the arguments in the proof of Theorem 5.1 only rely on Theorem 2.1. What our theorem really shows is that if $F$ is a flabby sheaf such that a quotient presheaf $F/G$ is a sheaf, then $F/G$ is flabby.

In order to be able to apply this result to the sheaf $C$, we need to recall two definitions from sheaf theory:

**Definition 5.4** Let $f : X \to Y$ be a continuous map between two topological spaces, and let $S$ be a sheaf on $Y$. Then the correspondence

$$X \supset U \to \text{ind lim} S(f(U))_{f(V)}$$

is a presheaf on $X$, whose associated sheaf will be called the inverse image of $S$ by $f$ and denoted by $f^{-1}S$.

**Definition 5.5** Let $f : X \to Y$ be as above and let $T$ be a sheaf on $X$. The sheaf defined on $Y$ by the correspondence

$$Y \supset V \to T(f^{-1}(V))$$

is called the direct image of $T$ by $f$ and denoted by $f_*T$. We can then look at the sheaf $C$ in a slightly different fashion: let, as before,

$$\pi : S^* \mathbf{H} \to \mathbf{H}$$

be the canonical projection; consider the subsheaf $G^*$ of $\pi^{-1}F$ defined by

$$G^*([U_1 \times (+\infty)] \cup [U_2 \times (-\infty)])$$

$$= \{ f \in F(U_1) : s.s.(f) \cap [U_1 \times (+\infty)] = \emptyset \}$$

$$\oplus \{ f \in F(U_2) : s.s.(f) \cap [U_2 \times (-\infty)] = \emptyset \}.$$ 

One can then immediately verify that

$$C = \frac{\pi^{-1}F}{G^*}.$$
and therefore, by definition, we have the following short exact sequence of sheaves on $S^*\hat{H}$:

$$0 \to \mathcal{G}^* \to \pi^{-1}\mathcal{F} \to \mathcal{C} \to 0,$$

which we now would like to project down to $\hat{H}$.

We finally have

**Theorem 5.2** For any $U \subseteq S^*\hat{H}$,

$$\mathcal{C}(U) = \frac{\pi^{-1}\mathcal{F}(U)}{\mathcal{G}^*(U)},$$

i.e. the short sequence

$$0 \to \mathcal{G}^*(U) \to \pi^{-1}\mathcal{F}(U) \to \mathcal{C}(U) \to 0$$

(5.3)

is exact.

**Proof** To prove the exactness of (5.3) it is clearly enough to prove that the map

$$\pi^{-1}\mathcal{F}(U) \to \mathcal{C}(U)$$

is onto. Since, as we mentioned earlier, $S^*\hat{H} = \hat{H} \amalg \hat{H}$, we will assume $U$ to be of the form

$$U = \Omega \times \{+\infty\}.$$

Let $\{\Omega_i\}$ be a locally finite covering of $\Omega$ such that a given $f \in \mathcal{C}(U)$ is locally defined as $f_j \mod \mathcal{G}^*(\Omega_j \times \{+\infty\})$ for some $f_j \in \mathcal{F}(\Omega_j)$. We want to show how we can construct a global section $f \in \mathcal{F}(\Omega)$. By the definition of $\mathcal{G}^*$, we have

$$f_j(x) \equiv b_+ F_j(x) \mod \mathcal{G}^*(\Omega_j \times \{+\infty\}).$$

Now the compatibility condition

$$f_{ij} = f_j|_{\Omega_i \cap \Omega_j} - f_i|_{\Omega_i \cap \Omega_j} \in \mathcal{G}^*(\Omega_i \cap \Omega_j \times \{+\infty\})$$

implies that

$$f_{ij} = b_+ F_j - b_+ F_i$$

can actually be regularly continued to $x_0 < 0$, and it is indeed a function in $\mathcal{G}$. Thus

$$\{f_{ij}\} \in Z^1(\Omega, \mathcal{G}).$$

By the same argument which we employed in Theorem 5.1, one can find a 0-cochain $g_i \in \mathcal{G}(\Omega_i)$ such that

$$f_{ij} = g_j - g_i.$$

Now we see that

$$f_i = b_+ F(x) - g_i \mod \mathcal{G}^*(\Omega_i \times \{+\infty\})$$

and so we can glue together the $f_j$ to obtain the desired $f \in \mathcal{F}(\Omega)$.

**Corollary 5.1** The sheaf $\mathcal{C}$ is flabby.
Proof This is an immediate consequence of the fact that, for any open set $U$, 
\[ C(U) = \pi^{-1}F(U) \]
and that $F$ (and $\pi^{-1}F$) are flabby sheaves.

Let us finally define, for any $\Omega$ open in $\mathbf{H}$, the spectral mapping 
\[ s(\pi) : F(\Omega) \rightarrow C(\pi^{-1}(\Omega)) \]
by the following diagram:

\[ F(\Omega) \rightarrow \pi^{-1}F(\pi^{-1}(\Omega)) = F(\Omega) \oplus F(\Omega) \rightarrow C(\pi^{-1}(\Omega)) \]
\[ f \rightarrow f \oplus f \rightarrow [f \bmod \mathbf{G}^*(\Omega \times \{+\infty\})] \oplus [f \bmod \mathbf{G}^*(\Omega \times \{-\infty\})] \]

Then we immediately have:

\begin{corollary}
The following sequences are exact:
\[ 0 \rightarrow \mathcal{G}(\Omega) \rightarrow F(\Omega) \rightarrow C(\pi^{-1}(\Omega)) \rightarrow 0 \]
\[ 0 \rightarrow \mathcal{G} \rightarrow F \rightarrow \pi_*\mathcal{G} \rightarrow 0. \]
\end{corollary}

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