Regular functions on a Clifford Algebra

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Abstract
In [3], [4], [5] the authors offered an alternative definition and theory of regularity for functions of a quaternionic or octonionic variable, inspired by an idea of Cullen [2]. This alternative theory is intriguing because it allows the study of natural power series (and polynomials) with quaternionic or octonionic coefficients, which is excluded when the Fueter approach is followed. In this paper the authors follow the same ideas to offer a new definition of regularity on the Clifford Algebra \( Cl(0,3) \). It turns out that this setting presents a family of new phenomena, which did not appear in the cases of quaternions and octonions. It is nevertheless possible to find a power series expansion for functions regular on an appropriate subset of \( Cl(0,3) \).

1 Introduction
In a recent paper, [7], the authors have shown how a new theory of power series with quaternionic and octonionic coefficients (see [3], [4], [5], [6]) can be used to provide a direct extension of Gauss’ renowned proof of the fundamental theorem of algebra to the case of polynomials with quaternionic (octonionic) coefficients.

In this article, we tackle the next natural setting, namely the case of functions defined on the Clifford Algebra \( Cl(0,3) \) with three generators. It becomes clear from the beginning, that this algebraic setting requires a different kind of approach, and we devote section two to the analysis of the kind of algebraic representations which are possible for the elements of \( Cl(0,3) \). This analysis allows the definition of a Cauchy-Riemann like operator, whose solutions we call regular functions.

In the last section of the paper, we show that these functions can in fact be represented as power series with Clifford coefficients.

One of the most interesting features of this new theory is the existence of a natural boundary for these functions. Indeed, it turns out that the notion of regularity can only be defined outside a codimension three closed set of \( Cl(0,3) \), in contrast with the simpler situation which occurs in the case of Hamilton and Cayley numbers. It would be interesting to study the boundary values of regular functions along this natural border.

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2 Algebraic properties of $Cl(0, 3)$

Let $Cl(0, 3)$ denote the real Clifford algebra of signature $(0, 3)$. This algebra can be defined as follows (see [1] for this and other related definitions): let $E = \{e_1, e_2, e_3\}$ be the canonical orthonormal basis for $\mathbb{R}^3$ with defining relations $e_ie_j + e_je_i = -2\delta_{ij}$. An element of the Clifford Algebra $Cl(0, 3)$ can be written in a unique way as

$$x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 + x_{123}e_1e_2e_3$$

where the coefficients $x_i, x_{ij}, x_{ijk}$ are real numbers. Thus, we see that $Cl(0, 3)$ is an eight dimensional real space, endowed with a natural multiplicative structure. Note that the square of each unit $e_i, e_ie_j$ is negative one, while the square of $e_1e_2e_3$ is one. For this reason, the element $e_1e_2e_3$ is often referred to as a pseudoscalar. Therefore it is appropriate to define the set of Clifford real numbers as $\mathcal{R} = \{x = x_0 + x_{123}e_1e_2e_3\}$ and the set of Clifford imaginary numbers as $\mathcal{I} = \{x = x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3\}$. With these definitions, we can decompose any element $x$ of $Cl(0, 3)$ as the sum of an element in $\mathcal{R}$, its real part $Re(x)$, and an element in $\mathcal{I}$, its imaginary part $Im(x)$.

By using the defining relations indicated above, one can easily construct the multiplication table for $Cl(0, 3)$, which will be useful in what follows to verify all the subsequent computations.

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Let $\mathbb{K}$ denote any of the algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}, Cl(0, 3)$. For each of these algebras, we can define the set of roots of $-1$ as $\mathbb{S}_\mathbb{K} = \{w \in \mathbb{K} : w^2 = -1\}$. A second set of interest is what we call the $\mathbb{S}_\mathbb{K}$-unit imaginary sphere, namely the set $\mathbb{U}_\mathbb{K} = \{w \in \mathbb{K} : Re(w) = 0, |Im(w)| = 1\}$. The previous papers ([4], [5], [6]) exploit the identity between $\mathbb{S}_\mathbb{K}$ and $\mathbb{U}_\mathbb{K}$ which holds for $\mathbb{K} = \mathbb{C}, \mathbb{H}, \mathbb{O}$, and can be easily verified. A first interesting phenomenon which occurs in the Clifford Algebra case is the fact that $\mathbb{S}_{Cl(0,3)}$ is properly contained in $\mathbb{U}_{Cl(0,3)}$, and in fact the following proposition holds.

**Proposition 1.** An element $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 + x_{123}e_1e_2e_3$ in $Cl(0, 3)$ belongs to $\mathbb{S}_{Cl(0,3)}$ if and only if it belongs to $\mathbb{U}_{Cl(0,3)}$ and its coordinates satisfy

$$x_1x_{23} - x_2x_{13} + x_3x_{12} = 0.$$

**Proof.** The square of $x$ is given by

$$x^2 = x_0^2 + x_{123}^2 - x_1^2 - x_2^2 - x_3^2 - x_{12}^2 - x_{13}^2 - x_{23}^2 + 2e_1(x_0x_1 - x_{23}x_{123}) + 2e_2(x_0x_2 + x_{13}x_{123}) + 2e_3(x_0x_3 - x_{12}x_{123}) + 2e_1e_2(x_0x_{12} - x_{3}x_{123}) + 2e_2e_3(x_0x_{13} + x_{2}x_{123}) + 2e_1e_3(x_0x_{23} - x_{1}x_{123}) + 2e_1e_2e_3(x_0x_{123} + x_1x_{23} - x_2x_{13} + x_3x_{12}).$$
By rewriting $x^2 = -1$ we obtain the system
\[
\begin{aligned}
x_0^2 + x_{123}^2 - x_1^2 - x_2^2 - x_3^2 - x_{12}^2 - x_{13}^2 - x_{23}^2 &= -1 \\
x_0x_1 - x_{23}x_{123} &= 0 \\
x_0x_2 + x_{13}x_{123} &= 0 \\
x_0x_3 - x_{12}x_{123} &= 0 \\
x_0x_{12} - x_3x_{123} &= 0 \\
x_0x_{13} + x_2x_{123} &= 0 \\
x_0x_{23} - x_1x_{123} &= 0 \\
x_0x_{123} + x_1x_{23} - x_2x_{13} + x_3x_{12} &= 0
\end{aligned}
\]

If we assume that $x_0 \neq 0$, we can solve equations two through seven in the system and obtain (with $\sigma = (i, j, k)$ being any permutation of $(1, 2, 3)$)
\[
x_i = \text{sgn}(\sigma) \frac{x_{jk}x_{123}}{x_0},
\]
and
\[
x_{jk} = \text{sgn}(\sigma) \frac{x_i x_{123}}{x_0}.
\]

These two sets of equations imply
\[
x_i = x_i \left(\frac{x_{123}}{x_0}\right)^2
\]
and
\[
x_{jk} = x_{jk} \left(\frac{x_{123}}{x_0}\right)^2.
\]

Note that if $x_0 \neq 0$, it is impossible for all the $x_i$ and all $x_{jk}$ to be zero, since the first equation of the system would give $x_0^2 + x_{123}^2 = -1$ which is a contradiction.

Therefore if $x_0 \neq 0$ there is at least one of the $x_i$ or $x_{jk}$ which is different from zero, and so the two sets of equations we have written imply that $x_0^2 = x_{123}^2$, and $x_{123} \neq 0$, and therefore $x_0^2 + x_{123}^2 = 2x_0^2$.

Assume $x_0 = x_{123}$ (the case $x_0 = -x_{123}$ is totally analogous). Then the system can be rewritten as
\[
\begin{aligned}
2x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_{12}^2 - x_{13}^2 - x_{23}^2 &= -1 \\
x_1 - x_{23} &= 0 \\
x_2 + x_{13} &= 0 \\
x_3 - x_{12} &= 0 \\
x_0^2 + x_1^2 + x_2^2 + x_3^2 &= 0
\end{aligned}
\]
which is a contradiction. Therefore $x_0 = 0$, which immediately implies $x_{123} = 0$, $x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 = 1$ and $x_1x_{23} - x_2x_{13} + x_3x_{12} = 0$.

This concludes the proof. □

**Remark 2.** The exactly same computation shows, on the other hand, that the set of $w \in Cl(0, 3)$ such that $w^2 = 1$ reduces to $w = \pm 1$ and $w = \pm e_1e_2e_3$. 

3
Proposition 3. Given an element \( x \) of the Clifford algebra as the sum of a Clifford real number and the product of an element of \( S \) by a Clifford real number, namely

\[
x = (\alpha + \beta e_1 e_2 e_3) + I(\gamma + \delta e_1 e_2 e_3)
\]

where \( \alpha, \beta, \gamma, \delta \) are real numbers and \( I \in S \).

We will see that this is not always possible, but that the set of exceptions is sufficiently small, in fact it has codimension 3 in \( \mathbb{R}^8 \). To obtain this result we need a few preliminary computations. First we note that if \( w = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_{12} e_1 e_2 + x_{13} e_1 e_3 + x_{23} e_2 e_3 \) is a Clifford imaginary number, and \( (\gamma + \delta e_1 e_2 e_3) \) is a Clifford real number, then their product is given by

\[
w(\gamma + \delta e_1 e_2 e_3) = e_1(\gamma x_1 - \delta x_{23}) + e_2(\gamma x_2 + \delta x_{13}) + e_3(\gamma x_3 - \delta x_1)
\]

\[+ e_1 e_2(\gamma x_{12} - \delta x_3) + e_1 e_3(\gamma x_{13} + \delta x_2) + e_2 e_3(\gamma x_{23} - \delta x_1). \tag{1}\]

If, on the other hand, we multiply two Clifford imaginary numbers \( u = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_{12} e_1 e_2 + x_{13} e_1 e_3 + x_{23} e_2 e_3 \) and \( v = y_1 e_1 + y_2 e_2 + y_3 e_3 + y_{12} e_1 e_2 + y_{13} e_1 e_3 + y_{23} e_2 e_3 \) we obtain

\[
wv = -x_1 y_1 - x_2 y_2 - x_3 y_3 - x_{12} y_{12} - x_{13} y_{13} - x_{23} y_{23} + e_1(x_{12} y_1 + x_{23} y_3 - x_{13} y_2 - x_{23} y_1)
\]

\[+ e_2(-x_{12} y_2 + x_3 y_2 - x_{13} y_3 + x_{23} y_1) + e_3(-x_{13} y_1 + x_2 y_3 + x_{12} y_2)
\]

\[+ e_1 e_2(x_{12} y_1 + x_{23} y_3 - x_{13} y_2 + x_{31} y_3) + e_1 e_3(x_{13} y_1 - x_{12} y_2 + x_{23} y_3)
\]

\[+ e_2 e_3(x_{23} y_3 - x_{31} y_2 + x_{13} y_1 - x_{12} y_3 + x_{23} y_1). \tag{2}\]

**Proposition 3.** Given an element \( w = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_{12} e_1 e_2 + x_{13} e_1 e_3 + x_{23} e_2 e_3 \) in \( \mathcal{I} \), there are two distinct real numbers \( a \) and \( b \), and an element \( I \in S \) such that \( w = I(a + be_1 e_2 e_3) \) if and only if \( (x_1, x_2, x_3) \neq \pm (x_{23}, -x_{13}, x_{12}) \).

**Proof.** Let \( \lambda = (x_1, x_2, x_3) \) and \( \mu = (x_{23}, -x_{13}, x_{12}) \). To prove the result, we seek under which conditions it is possible to find two real numbers \( \gamma \) and \( \delta \) such that \( w(\gamma + \delta e_1 e_2 e_3) \) belongs to \( S \) and \( (\gamma + \delta e_1 e_2 e_3) \) is invertible in \( \mathcal{R} \). From (1), and by Proposition 1, this is equivalent to require, for \( \epsilon = \frac{\gamma}{\delta} \),

\[
(\lambda \mu) \epsilon^2 - (|\lambda|^2 + |\mu|^2) \epsilon + \lambda \mu = 0. \tag{3}
\]

The discriminant of equation (3) is

\[
\Delta = (|\lambda|^2 - |\mu|^2)^2 \geq 0
\]

and therefore the equation is always solvable in \( \epsilon \). Note that each solution for \( \epsilon \) gives (up to a sign) only two solutions for the pair \( (\gamma, \delta) \) because of the requirement that the product \( w(\gamma + \delta e_1 e_2 e_3) \) has unit length. We now want to see whether any of the solutions for \( \gamma + \delta e_1 e_2 e_3 \) we have just obtained is invertible. To this aim let us try to solve

\[
(\gamma + \delta e_1 e_2 e_3)(a + be_1 e_2 e_3) = 1.
\]

This equation has a solution in \( (a, b) \) if and only if \( \gamma^2 \neq \delta^2 \), i.e. \( \epsilon \neq \pm 1 \). The explicit form of the solution for \( \epsilon \) readily shows that the only case in which \( \epsilon = \pm 1 \) is in fact when \( \lambda = \pm \mu \). \( \square \)
Remark 4. A close inspection of the proof of the previous proposition actually shows that, for a given element \( w = x_1e_1 + x_2e_2 + x_3e_3 + x_12e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 \) in \( I \), with the condition \((x_1, x_2, x_3) \neq (x_{23}, -x_{13}, x_{12})\), there are (up to a sign) actually two imaginary units \( I \) and \( J \) and two possible pairs \((a, b)\) satisfying the conclusion of the previous propositions. In fact if \( w = I(a + be_1e_2e_3) \), then it is also true that there exists \( J \in \mathbb{S} \) such that \( w = I(a + be_1e_2e_3) = J(b + ae_1e_2e_3) \). The computations show that if \( I = i_1e_1 + i_2e_2 + i_3e_3 + i_{12}e_1e_2 + i_{13}e_1e_3 + i_{23}e_2e_3 \) and \( J = j_1e_1 + j_2e_2 + j_3e_3 + j_{12}e_1e_2 + j_{13}e_1e_3 + j_{23}e_2e_3 \), then \( I \) and \( J \) are orthogonal and their coordinates are related by the two systems

\[
\begin{cases}
  i_1 = -j_{23} & i_{12} = -j_3 \\
  i_2 = j_{13} & i_{13} = j_2 \\
  i_3 = -j_{12} & i_{23} = -j_1.
\end{cases}
\]

In fact, it is easy to show that \( J = Ie_1e_2e_3 \) and that therefore

\[
I(a + be_1e_2e_3) = Ie_1e_2e_3e_1e_2e_3(a + be_1e_2e_3) = J(b + ae_1e_2e_3).
\]

In order to develop a function theory on \( Cl(0, 3) \), we need to show that, given \( I \in \mathbb{S} \), it is possible to start with \( 1, e_1e_2e_3, I, Ie_1e_2e_3 \), choose \( J \in \mathbb{S} \) and complete a basis for \( Cl(0, 3) \) through the vectors \( J, Je_1e_2e_3, IJ, IJe_1e_2e_3 \). This will allow us immediately to show that every element \( x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_{12}e_1e_2 + x_{13}e_1e_3 + x_{23}e_2e_3 + x_{123}e_1e_2e_3 \in Cl(0, 3) \) can be represented as

\[
x = X_1 + X_2e_1e_2e_3 + X_3I + X_4Ie_1e_2e_3 + X_5J + X_6Je_1e_2e_3 + X_7IJ + X_8IJe_1e_2e_3
\]

and, since \( Je_1e_2e_3 = e_1e_2e_3J \), as

\[
x = (X_1 + X_2e_1e_2e_3) + I(X_3 + X_4e_1e_2e_3) + [(X_5 + X_6e_1e_2e_3) + I(X_7 + X_8e_1e_2e_3)]J.
\]

Proposition 5. Given \( I \in \mathbb{S} \), it is possible to choose \( J \in \mathbb{S} \) such that \( J \) is perpendicular to \( I \) and to \( Ie_1e_2e_3 \). Moreover, for such a \( J \), the set \( \mathcal{B} = \{1, e_1e_2e_3, I, Ie_1e_2e_3, J, Je_1e_2e_3, IJ, IJe_1e_2e_3\} \) is a basis for \( Cl(0, 3) \), whose elements (except for \( 1 \) and \( e_1e_2e_3 \)) are in \( \mathbb{S} \).

Proof. The existence of \( J \) with the prescribed perpendicularity is an immediate dimensionality consequence. Noticing that two perpendicular elements in \( \mathbb{S} \) anticommute, the fact that all the elements of the set \( \mathcal{B} \) are mutually perpendicular is an immediate consequence of direct computations taking into account Proposition 1, and equations (1) and (2).

\[
2D_1f_I = (d_{12} + Id_{34})f_I = 0,
\]

where \( d_{ij} = d_{i,t_j} \) indicates the differential with respect to the variables \( t_i \) and \( t_j \).

3 Regular functions and power series expansion

We will denote by \( U \) the set of all Clifford numbers \( x \) in \( Cl(0, 3) \) such that \( x = 0 \) or \((x_1, x_2, x_3) \neq \pm(x_{23}, -x_{13}, x_{12})\). As we have already noticed, by its definition, \( Cl(0, 3) \setminus U \) has codimension 3 in \( \mathbb{R}^8 \).

Definition 6. Let \( \Omega \) be a domain in \( U \). A real differentiable function \( f : \Omega \rightarrow Cl(0, 3) \) is said to be regular if, for every \( I \in \mathbb{S} \), its restriction \( f_I \) to the four dimensional Clifford plane \( L_I = \mathcal{R} + \mathcal{IR} = \{(t_1 + t_2e_1e_2e_3) + I(t_3 + t_4e_1e_2e_3)\} \) passing through the origin and containing \( 1 \) and \( I \) satisfies, on \( \Omega \cap L_I \), the system

\[
2D_1f_I = (d_{12} + Id_{34})f_I = 0,
\]

where \( d_{ij} = d_{i,t_j} \) indicates the differential with respect to the variables \( t_i \) and \( t_j \).
Contrary to what happens in the Hamilton and in the Cayley cases, the choice of the unit \( I \in \mathbb{S} \) is not unique, once a point \( x \in U \) is given, as we have noticed in Remark 4. Thus we need to analyze what is the implication of the two different representations which one can have for a point \( x \in U \). Let \( x \) be a Clifford number in \( U \) which admits two representations, say

\[
x = x_I = (t_1 + t_2e_1e_2e_3) + I(t_3 + t_4e_1e_2e_3)
\]
as well as

\[
x = x_K = (t_1 + t_2e_1e_2e_3) + K(t_3 + t_3e_1e_2e_3).
\]

Note that by rearranging we can write \( x = x_I = x_K \) in a more explicit way as

\[
x_I = (t_1 + It_3) + (t_2 + It_4)e_1e_2e_3
\]
and

\[
x_K = (t_1 + Kt_4) + (t_2 + Kt_3)e_1e_2e_3.
\]

By dimensionality, we can find another unit \( J \in \mathbb{S} \) which allows, in the sense of Proposition 5, to complete both the basis generated by \( I \) and the basis generated by \( K \). Then for the definition to be consistent we need to ask that

\[
2D_{x_I}f_I = (d_{12} + Id_{34})f_I(x_I) = 2D_Kf_K = (d_{12} + Kd_{43})f_K(x_K) = 0.
\]

To understand the consequence of this request we need to express the values of \( f_I \) and \( f_K \) in terms of the basis of \( Cl(0,3) \). To this purpose, we write

\[
f_I(x_I) = (f_{00} + f_{01}e_1e_2e_3) + I(f_{10} + f_{11}e_1e_2e_3) + [(g_{00} + g_{01}e_1e_2e_3) + I(g_{10} + g_{11}e_1e_2e_3)]J =
\]

\[
(f_{00} + If_{10}) + (f_{01} + If_{11})e_1e_2e_3 + [(g_{00} + Ig_{10}) + (g_{01} + Ig_{11})e_1e_2e_3]J =
\]

\[
F_0 + F_1e_1e_2e_3 + (G_0 + G_1e_1e_2e_3)J.
\]

Similarly, taking into account Remark 4 we have

\[
f_K(x_K) = (f_{00} + f_{01}e_1e_2e_3) + K(f_{11} + f_{10}e_1e_2e_3) + [(g_{00} + g_{01}e_1e_2e_3) + K(g_{11} + g_{10}e_1e_2e_3)]J =
\]

\[
(f_{00} + Kf_{11}) + (f_{01} + Kf_{10})e_1e_2e_3 + [(g_{00} + Kg_{11}) + (g_{01} + Kg_{10})e_1e_2e_3]J =
\]

\[
M_0 + M_1e_1e_2e_3 + (N_0 + N_1e_1e_2e_3)J.
\]

Thus, Definition 6 is equivalent to the following four two-dimensional Cauchy-Riemann systems:

\[
\begin{align*}
\begin{cases}
d_{12}f_{00} = d_{34}f_{10} \\
d_{12}f_{10} = -d_{34}f_{00}
\end{cases}
\end{align*}
\begin{align*}
\begin{cases}
d_{12}f_{01} = d_{34}f_{11} \\
d_{12}f_{11} = -d_{34}f_{01}
\end{cases}
\end{align*}
\begin{align*}
\begin{cases}
d_{12}f_{00} = d_{43}f_{11} \\
d_{12}f_{11} = -d_{43}f_{00}
\end{cases}
\end{align*}
\]
The solutions of this system have a very nice geometrical interpretation in terms of the bases defined previously.

**Theorem 7.** Let $\Omega$ be a domain in $U$, let

$$f = F_0 + F_1 e_1 e_2 e_3 + (G_0 + G_1 e_1 e_2 e_3) J = M_0 + M_1 e_1 e_2 e_3 + (N_0 + N_1 e_1 e_2 e_3) J$$

be differentiable on $U$, and let

$$z_1 = t_1 + It_3, z_2 = t_2 + It_4, w_3 = t_1 + Kt_3, w_2 = t_2 + Kt_4.$$

Then $f$ is regular in $\Omega$ if and only if $F_0, F_1, G_0, G_1$ are holomorphic in $(z_1, z_2)$, and $M_0, M_1, N_0, N_1$ are holomorphic in $(w_1, w_2)$.

**Proof.** This is an immediate consequence of rewriting the four two-dimensional Cauchy-Riemann systems in terms of their components. \qed

It may be worthwhile to show that in fact there are non-trivial examples of regular functions in this sense. For example, with the same notations as above, we could consider

$$\begin{cases}
  f_{00} = t_1 t_2 - t_3 t_4 \\
  f_{10} = t_2 t_3 + t_1 t_4
\end{cases}$$

and

$$\begin{cases}
  f_{01} = \frac{1}{2}(t_1^2 + t_2^2 - t_3^2 - t_4^2) \\
  f_{11} = t_1 t_3 + t_2 t_4
\end{cases}$$

This gives $F_0 = z_1 z_2, F_1 = \frac{1}{2}(z_1^2 + z_2^2), M_0 = \frac{1}{2}(w_1^2 + w_2^2), M_1 = w_1 w_2$, which satisfy the conditions of the theorem, and offer an example of a regular function.

We now prove that regular functions can be expressed in power series.

**Theorem 8.** Let $f$ be regular function in $U$. For an arbitrary point $w = z_1 + z_2 e_1 e_2 e_3$ in $U$, there are Clifford numbers $a_{mn}$ such that

$$f(z_1 + z_2 e_1 e_2 e_3) = \sum_{m,n \in \mathbb{N}} z_1^m z_2^n a_{mn}$$

where

$$a_{mn} = \frac{\partial^{m+n} f(0)}{\partial z_1^m \partial z_2^n}.$$

**Proof.** Theorem 7 shows that the function $f$ can be written as

$$f(z_1 + z_2 e_1 e_2 e_3) = F_0(z_1 + z_2 e_1 e_2 e_3) + F_1(z_1 + z_2 e_1 e_2 e_3) + (G_0(z_1 + z_2 e_1 e_2 e_3) + G_1(z_1 + z_2 e_1 e_2 e_3)) J,$$

with $F_0, F_1, G_0, G_1$ holomorphic. Thus all these functions admit a power series representation and we have

$$f(z_1 + z_2 e_1 e_2 e_3) = \sum_{m,n \in \mathbb{N}} z_1^m z_2^n \frac{\partial^{m+n} F_0(0)}{\partial t_1^m \partial t_2^n} + \sum_{m,n \in \mathbb{N}} z_1^m z_2^n \frac{\partial^{m+n} F_1(0)}{\partial t_1^m \partial t_2^n} e_1 e_2 e_3 +$$

$$\sum_{m,n \in \mathbb{N}} z_1^m z_2^n \frac{\partial^{m+n} G_0(0)}{\partial t_1^m \partial t_2^n} J + \sum_{m,n \in \mathbb{N}} z_1^m z_2^n \frac{\partial^{m+n} G_1(0)}{\partial t_1^m \partial t_2^n} e_1 e_2 e_3 J = \sum_{m,n \in \mathbb{N}} z_1^m z_2^n \frac{\partial^{m+n} f(0)}{\partial t_1^m \partial t_2^n}.$$

Since the coefficients of these series do not depend on the choice of $I \in \mathbb{S}$, this concludes the proof. \qed
Note that the series which appear in this last theorem are defined for all values of $z_1$ and $z_2$ (and for all $I \in S$), but only express regular functions defined on $U$.

References


